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The Applicability of Mathematics as a Scientific and a Logical Problem[†]

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This paper explores how to explain the applicability of classical mathematics to the physical world in a radically naturalistic and nominalistic philosophy of mathematics. The applicability claim is first formulated as an ordinary scientific assertion about natural regularity in a class of natural phenomena and then turned into a logical problem by some scientific simplification and abstraction. I argue that there are some genuine logical puzzles regarding applicability and no current philosophy of mathematics has resolved these puzzles. Then I introduce a plan for resolving the logical puzzles of applicability.

1. Introduction

Any philosophy of mathematics must be able to explain the applicability of mathematics to the physical world. This study of applicability belongs to a research project pursuing a *radically naturalistic* approach to philosophy of mathematics. I take it that naturalism implies that human cognitive subjects are human brains as physical systems. For philosophy of mathematics, I believe this implies nominalism, because a completely naturalistic description of the cognitive activities (including mathematical practices) of a *brain* will describe neural activities inside the brain and their physical interactions with the environment *only* and will not mention what abstract objects the brain refers to.[1] If that is correct, it implies that what

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[1] See [Ye, online-b] for a full argument. Here I will just briefly respond to one natural suspicion: in describing the activities of a brain, we still have to refer to abstract mathematical objects. This is a misunderstanding. Suppose a brain B is using a mathematical term T. In describing B, I describe T as a neural structure inside B and describe the physical interactions between T and other neural structures inside B and physical things in the environment, but I do not need to mention which abstract objects that neural structure T 'refers to'. Then, mathematical terms as neural structures inside *my* brain in describing B do not 'refer to' anything either, because my brain is just like B.

really exist in human mathematical practices are *only* human brains, neural activities inside brains, and their physical interactions with other physical things in environments (when the brains are applying mathematics to those physical things). A study of human mathematical practices should then be a branch of science literally, since it aims at studying a class of natural phenomena. This is what I mean by 'a radically naturalistic approach to philosophy of mathematics'. There have been several versions of naturalism in philosophy of mathematics [Maddy, 2005]. I will call my version 'radical naturalism' to emphasize its radical nature. See [Ye, online-a] for an introduction to the entire project.

In this paper, I will first show that, under radical naturalism, the applicability of mathematics means some natural regularity in a class of natural phenomena, and an explanation of applicability should then be a scientific explanation of some natural phenomena. That is, the applicability problem is naturalized and becomes a scientific problem. There are many aspects involved in human mathematical practices and applications, from the psychological aspect to historical and sociological ones. As a logician and philosopher, I am interested only in the logical and philosophical aspects and by 'applicability' I mean the philosophical and logical reasons for applicability. I will show that, by some *scientific* abstraction and simplification to ignore physical, psychological, and other details (just as we ordinarily do in many branches of science), the problem of applicability becomes a *logical* problem. I will then argue that realism in philosophy of mathematics has not offered any logically clear explanation of applicability, even if we agree that mathematical entities exist and mathematical theorems are true of them. I will also point out that current anti-realistic philosophies of mathematics have not explained applicability either. In particular, none of them has resolved some genuine *logical* puzzles regarding the role of infinity in current well-established scientific theories about strictly finite physical phenomena above the Planck scale. Finally, I will sketch a plan for resolving these logical puzzles and explaining applicability by showing that infinity is in principle dispensable for applications to finite things. Implementing the plan requires significant technical work in logic and mathematics. The work is still in progress, but what has been achieved so far seems to indicate that the plan is feasible. The details of the technical work done so far are reported in a monograph available online [Ye, online-d]. This paper introduces the philosophical aspect of the plan. Another paper [Ye, online-e] summarizes the technical work in that monograph.

2. Basic Assumptions about Human Cognitive Architecture

To explore such a radically naturalistic description of human mathematical practices, we need some basic assumptions about human cognitive architecture. So far we know very little about human cognitive architecture. However, since our interest is in only the philosophical and logical aspects of human mathematical practices, we can ignore psychological details and rely on a greatly simplified model of human cognitive architecture, as long as we have reasons to believe that these simplifications will not invalidate our answers to the *philosophical and logical* questions regarding human mathematical practices. I will give a summary of the assumptions taken by my approach. These assumptions are well known in the philosophy of mind and cognitive science, but a brief summary seems necessary, since some philosophers of mathematics may not be very familiar with the relevant terminologies.

First, I will assume the *Representational Theory of Mind*, which means that brains create inner representations *realized as neural structures* in them. Brains associate linguistic expressions with inner representations in order to communicate them. We say that linguistic expressions *express* inner representations.

Some inner representations are *concepts*. Note that concepts here are concrete neural structures in individual brains, not the Fregean concepts or senses as public and abstract entities. Some concepts *represent* physical objects or their properties. For instance, a concept RABBIT in someone's brain *expressed* by the word 'rabbit' may *represent* rabbits. I will call these *realistic concepts*. This representation relation is a sort of physical connection between a neural structure in a brain and other physical entities or properties outside the brain. Characterizing this representation relation in naturalistic and scientific terms (*i.e.*, without using intentional or semantic terms such as 'represent', 'mean', 'refer to', *etc.*) is called 'naturalizing content' in philosophy of mind. I will assume here that this representation relation *can* be naturalized.[2] This also means naturalizing semantic normativity in the representation relation, or accounting for representational errors under naturalism. Some other concepts do not represent anything directly. They have more flexible and abstract cognitive functions inside a brain, and they can connect with physical things outside the brain *indirectly*. I will call these *abstract concepts*. Mathematical concepts are abstract concepts. For instance, a mathematical concept expressed by the word '2' in a brain can combine with a realistic concept RABBIT to form a composite concept 2-RABBIT, which does represent physical things outside the brain directly. Similarly, in applying a geometrical theory to physical space, a mathematical concept POINT in the geometrical theory in a brain is translated into a realistic concept representing small physical-space regions directly. Such translations of abstract concepts into realistic concepts

[2] See [Adams, 2003] and [Neander, 2004] for surveys, and see [Ye, online-c] for a new approach.

are neural processes in the brain. Abstract concepts are abstract representational tools (as neural structures) inside brains.

Thoughts are another type of inner representation and are typically expressed by declarative sentences. For instance, a simple thought expressed by 'rabbits are animals' in a brain is composed of two concepts RABBIT and ANIMAL in that brain (in some way as a neural structure). This thought is *true in the naturalized sense* if the entities represented by RABBIT are among the entities represented by ANIMAL. Since the representation relation for realistic concepts is a naturalized relation, this is *naturalized truth* and is ultimately a physical connection between neural structures and other physical things as well. Note that this naturalized 'truth' applies only to thoughts composed of realistic concepts, which are *realistic thoughts*. Thoughts composed of abstract concepts do not represent any states of affairs directly. They are *abstract thoughts*. They have more flexible and abstract cognitive functions inside a brain, and they can similarly connect with physical things outside the brain indirectly. For instance, an abstract thought in a geometrical theory in a brain is translated into a realistic thought about physical space when the geometrical theory is applied to physical space, and then that realistic thought can connect with physical space by the naturalized representation relation.[3] There are also logically composite thoughts composed of other thoughts and *logical concepts*. I will assume that naturalized truth for thoughts composed of *realistic* simple thoughts will respect common logical rules. For instance, a thought 'p AND q' will be true just in case both the thoughts p and q are true. Note that, in naturalism, we do not intend to offer any foundational justification of logical truths or any non-circular definition of logical constants. We simply try to describe how human brains work, based on all our scientific knowledge, including our logical knowledge.[4]

A *logical-inference rule* is an inference-process pattern in brains, which produces a thought in some format as the conclusion from other thoughts in some formats as the premises. As a *logical*-inference pattern, we fix logical concepts in the pattern and consider other realistic or abstract concepts as variable. Therefore, a realistic thought may share the same format as an abstract thought, and an inference-process instance involving abstract thoughts can share the same inference pattern with an inference-process instance consisting exclusively of realistic thoughts. A logical-inference rule is *valid in the naturalized sense* if for any inference-process instance with that pattern and with realistic thoughts as the premises and conclusion, whenever the premises are true in the naturalized sense, the conclusion is

[3] See [Ye, online-f] for more details on the cognitive functions of mathematical concepts and thoughts.

[4] This is the stance of methodological naturalism (*e.g.*, [Maddy, 2007]. [Ye, online-g] discusses the status of logical knowledge under radical naturalism.

also true in the naturalized sense. Therefore, this is a naturalistic notion as well. An assertion about the naturalized validity of a logical-inference rule is an assertion about some regularity in a class of natural processes. Note that while a logical-inference pattern may apply to abstract thoughts as well as realistic thoughts, naturalized validity is characterized by the effects on only realistic thoughts, because naturalized truth is meaningful only for realistic thoughts. Moreover, note that a brain may frequently conduct logical inferences that are not valid according to this characterization. Therefore, there is normativity in naturalized logical validity. It is naturalized normativity coming from naturalized semantic normativity in the representation relation for realistic concepts.

All logical rules in classical first-order logic are valid in this naturalized sense. Again, remember that we are not offering any foundational justification for the validity of logical rules. We reach this conclusion based on our scientific knowledge. Some of these logical rules are *a priori and necessary in a naturalized sense*. That is, as a result of evolution, human brains have an innate cognitive architecture adapted to human environments so that some patterns of inference are universally valid *and* a human brain has the innate tendency to accept these rules after a normal maturation and learning process.[5] On the other hand, if there are only finitely many physical objects in the universe, then, for some numerical expression N, it may happen that all *realistic* thought instances of the format 'there are only N Ps' (for a predicate variable P) are as a matter of fact universally true. Then, this becomes a logically valid thought pattern in this naturalized sense.[6] However, this thought pattern is not *a priori* and necessary in the above naturalized sense, because brains do not have any innate tendency (as a result of evolution) to accept it (after a normal maturation process). It seems that traditional logical truths are naturalistically valid thought patterns that are also *a priori* and necessary in the naturalized sense (and are therefore knowable to brains), together with idealizations of these (for instance, by ignoring any limitation on the complexity of thoughts that could be produced by a brain).

Finally, some philosophers (*e.g.*, some connectionists) may deny that there is anything like a concept or thought inside the brain.[7] However, it is a fact that humans can do symbolic inferences. Therefore, for our purpose here, we can ignore psychological details and consider a bigger physical system consisting of a brain together with the words produced by that brain. We can view mathematical applications as interactions between this bigger system and its environments. Then, we can take those linguistic

[5] See [Ye, online-g] for a discussion on the apriority and necessity of logic under naturalism.

[6] I would like to thank a referee for raising this question.

[7] I would like to thank a referee for raising this issue.

expressions inside such a bigger 'brain' as concepts and thoughts. So, it seems harmless to assume that there *are* concepts and thoughts inside the brain.

3. Characterizing the Problem of Applicability

In a typical mathematical application, a brain starts with some realistic thoughts representing observed data about some physical entities. These are the *realistic premises* of the application. Then, the brain chooses a collection of mathematical concepts and thoughts for modeling those physical entities. The brain translates realistic premises into abstract mathematical thoughts by some *bridging postulations* of the application, which are thoughts connecting realistic concepts and abstract concepts. The results are *mathematical representations of realistic premises*. The brain may directly adopt some *mathematical premises* of the application for representing regularities among physical things. For instance, a mathematical premise can be a differential equation representing changes in the temperature of an object. Then, by a mathematical proof, the brain draws a *mathematical conclusion*. Finally, based on bridging postulations again, the brain translates the mathematical conclusion back into a realistic thought as the *realistic conclusion of the application*. The entire application process in the brain is then an inferential process

from the realistic premises, bridging postulations, and mathematical premises of the application, together with some pure mathematical theorems, to a realistic conclusion of the application. (1)

In this inference process on thoughts as neural structures in a brain, only the realistic premises and conclusion at the beginning and end can bear the naturalized 'true' relation with physical things outside the brain.

Explaining the applicability of mathematics then means explaining why the realistic conclusion is true (in the naturalized sense) in ordinary scientifically valid mathematical applications. In other words, there is a naturalistic 'true' (relational) property for realistic thoughts (as neural structures). We want to know why that property is present at the end of a process in a brain as in (1), in those instances of scientifically valid mathematical applications, while that property is not present at the beginning of the process and is not preserved at the intermediate stages (because it is not applicable for the mathematical premises at the beginning, nor for abstract mathematical thoughts in the intermediate stages). This is similar to explaining why a physical property, for instance, a property about mass or energy, is present at the end of a physical process, when it is not present at the beginning nor preserved at the intermediate stages. Therefore, this is a completely scientific question, not a question for metaphysical speculations. The applicability of mathematics is thus *naturalized*. Moreover, we must remember that under naturalism an explanation of applicability is not meant to be a foundational justification of applicability.

We want to ignore details as much as we can in studying the problem of applicability. In particular, we can distinguish the logical aspect from the psychological aspect. The psychological aspect concerns the psychological mechanisms involved in the mathematical activities of the brain. For instance, how do brains invent and operate on mathematical concepts, and what human cognitive architecture enables brains to do these? [Lakoff and Núñez, 2000] is an example of such research. In contrast, the logical aspect is concerned mainly with the logical structures of mathematical concepts, thoughts, and inference patterns in brains, and with how these structures allow finally producing a literally true realistic conclusion about physical things in an application scenario. In studying this logical aspect of mathematical applications, we can abstract away psychological and other details. For instance, as an approximation, we can assume that the structures of inner representations are just the syntactical structures of the linguistic expressions expressing them. Then, instead of talking about concepts and thoughts as inner representations realized as neural structures in brains, we can talk about linguistic expressions, as if those words and sentences were themselves in the brains. As a further simplification, I will assume that inner representations are syntactical entities in a language with a clearly defined basic vocabulary and syntax, such as a first-order language. Then, concepts are terms in the language, and thoughts are sentences there, and inference processes are just syntactical inferences on sentences. These assumptions appear reasonable for our specific purpose here. They amount to assuming that only the structures that are encoded in these syntactical entities are really relevant for explaining, from the logical and philosophical point of view, how an application finally produces a literally true realistic conclusion about physical entities. Other details beyond those structures, psychological or otherwise, are not really relevant.

We can distinguish between two vocabularies in the language. The realistic vocabulary is for expressing realistic concepts and thoughts. They constitute a sub-language L_r. The abstract vocabulary is for expressing abstract mathematical concepts and thoughts. They constitute another sub-language L_m. Terms and sentences in L_r are realistic terms and sentences, and those in L_m are abstract terms and sentences. Bridging sentences will use both vocabularies. Realistic terms and predicates in L_r have *fixed* semantic values consisting of physical entities or their properties in the real world, based on the naturalized representation relation. We will ignore the details in that naturalized representation relation and treat that relation as a satisfaction relation between a formal language and a semantic model consisting of physical entities. Therefore, L_r has a fixed semantic model M_r consisting of real physical entities. Let Γ_r be the collection of realistic premises, and let Γ_m be the collection of mathematical premises, including the premises expressing physics laws in a specific application instance and the mathematical axioms of classical mathematics, and let Γ_b be the collection of bridging postulations. An application is then a purely logical inference

$$\Gamma_r \cup \Gamma_m \cup \Gamma_b \vdash \varphi$$

from these premises to a realistic sentence φ in L_r as the realistic conclusion. We may assume that $M_r \models \Gamma_r$ when this application is scientifically valid. However, $M_r \models \Gamma_m \cup \Gamma_b$ does not hold, since the semantic model M_r consists of only physical entities. Then, the applicability problem becomes this logical problem:

> In a scientifically valid application, assuming that $M_r \models \Gamma_r$, why does $\Gamma_r \cup \Gamma_m \cup \Gamma_b \vdash \varphi$ imply $M_r \models \varphi$, for φ that is in the language L_r and is scientifically meaningful?

4. The Logical Puzzles of Applicability

It is sometimes claimed that realism in philosophy of mathematics has a ready explanation for the applicability of mathematics and that this is the advantage of realism over anti-realism. A realistic explanation claims that the conclusion in an inference process (1) above is true because all the premises there are true (although some of them are 'true of abstract mathematical entities'), and because the inference steps there preserve truth. Under naturalism, this means that

(i) there is a property 'true' that is applicable to both abstract thoughts and realistic thoughts in brains, and
(ii) this 'true' property is consistent with the naturalized 'true' property for realistic thoughts, and
(iii) this 'true' property is owned by all thoughts in the inference process (1) above.

From the naturalistic point of view, there are two problems with this alleged explanation; one is philosophical and the other is logical and technical. The philosophical problem is that realists have not offered any *naturalistic* characterization of the alleged 'true' property for *abstract* thoughts in brains satisfying the conditions (i)–(iii) above. I will not discuss the details of this problem here because I want to focus on the problem of applicability common to both realism and anti-realism.[8]

[8] Doubts about realistic truth are well-known, but radical naturalism does offer something new. For instance, if we really try to characterize what it is for an abstract thought as a neural structure in a brain to be 'true', we will quickly realize that we are referring *only*

The logical and technical problem is that, even if we agree that abstract mathematical objects exist and mathematical theorems are true of them, in many cases there is still no clear logical picture of why the conclusion drawn in an application is true of *physical* things. This is because of a well-known feature of applying infinite mathematics to the physical world: the physical things we deal with in current *well-established* scientific theories are strictly finite, from the Planck scale (about $10^{-35}m$, $10^{-45}s$, *etc.*) to the cosmological scale; infinite mathematical models are only 'approximations' to finite physical things in these applications and the logic of these 'approximations' is sometimes unclear; as a result, the premises of an application are sometimes not literally true and the application is not a simple case of valid logical deductions from literally true premises to a literally true conclusion. Ideally, a logically clear explanation of applicability should identify what literally true premises an application really assumes and then demonstrate how the conclusion drawn in the application logically follows from those literally true premises.[9] So far realists have not given this. I am not saying that this is impossible. On the contrary, I believe that it *is* possible, but in doing so, we may in the end find out that our conclusions about finite physical things logically and indispensably depend *only* on literally true premises about finite physical things. That is, mathematical theorems about infinite mathematical models are not really among the *logically minimum* premises required to derive our conclusions about finite physical things above the Planck scale in current well-established scientific theories.

For instance, consider the case of using a continuous model to simulate the motion of a fluid consisting of discrete particles. A mathematical premise here may claim that the mathematical model satisfies some differential equation. This comes from applying laws of physics to the continuous model, pretending that the mass in the fluid is distributed continuously. Then, our conclusions about those discrete particles in the fluid appear to depend on mathematical theorems about that continuous model and depend on the hypothesis that the model 'approximately simulates' the fluid. However, physicists certainly believe that laws of physics about collisions between those discrete particles in the fluid (and laws of physics about electro-magnetic force which finally account for the collision force) are the true fundamental physical premises that really imply our realistic conclusions about the fluid. Physicists do rely on experiments to confirm that a continuous mathematical model works fine for modeling the fluid,

to neural structures and their physical connections with other physical things, and alleged abstract objects never appear. See fn 1 and [Ye, online-b].

[9] Many philosophers have observed that mathematical models are not exactly true of physical things in the applications (*e.g.*, [Maddy, 1997]), but I take this to be a genuine logical puzzle to be resolved by logicians.

that is, to confirm that the model does 'approximately simulate' the fluid. However, they do not consider this to be discovering a new fundamental physical law of nature. They believe that this is only using experiments to confirm a simplified computation method. Our physical conclusions about the fluid should *in principle* follow from the fundamental physical laws (and observational data) about those discrete particles *alone*. For instance, if we have a gigantic computer that can simulate the motion of each particle directly by computing the forces exerted on each particle from all other particles (and from gravitation), then we will have a *literally more accurate* description of the motion of the fluid. This description will refer only to those discrete particles and their physical properties, and it will not assume infinity, continuity, or any abstract mathematical objects.[10] A physical conclusion about those discrete particles will then follow from fundamental physical laws and other observational data about those particles alone. That is, a derivation of a conclusion will be a series of valid logical deductions from literally true premises about finite physical things alone to a literally true conclusion about them. This suggests that the same should be true for any physically valid conclusion about the fluid drawn by applying that continuous mathematical model: the conclusion does not really depend on mathematical theorems about the continuous model, although applying the model simplifies our work. This example is not peculiar. Other applications of mathematics in current well-established theories may be similar, since they similarly describe only discrete things above the Planck scale. For instance, the standard mathematical formalism of classical quantum mechanics appears to refer to infinite mathematical entities such as wave functions. However, considering the fact that it is also accurate only above the Planck scale, our physical intuition seems to be that it is similar to using continuous models to simulate fluids. For instance, it is perhaps possible to discretize wave functions and Schrödinger's equation, and then, with a hypothetical gigantic computer, we can perhaps similarly simulate a system of quantum particles.

I admit that it is still unclear if this idea is valid. More technical work is required to clarify it. Moreover, this is certainly not an *a priori* philosophical argument. It is merely a vague intuition based on the experience of studying physics. However, it does suggest that there are some genuine *logical* questions regarding the applicability of classical infinite mathematics in current scientific theories about natural phenomena above the Planck scale. What are the logically minimum premises for deriving our scientific conclusions in these theories? Are mathematical theorems about infinite mathematical entities really among the logically minimum premises? From

[10] Remember that all physical quantities are meaningful only up to some finite precision. They can all be represented by numerals in a computer.

the logical point of view, why can infinite mathematical models, which do not represent finite physical things exactly, help to derive literal truths about finite physical things above the Planck scale? These are *the logical puzzles of applicability*. Realism has not resolved these puzzles yet. The observations above also suggest that an answer to these questions can perhaps be achieved by eliminating infinity and transforming the applications of infinite mathematics into logically valid deductions from literally true premises about finite physical things *alone* to literally true conclusions about them. If this is indeed the case, we will have a logically plain demonstration of applicability.

On the other hand, current anti-realistic philosophies of mathematics have not resolved these logical puzzles of applicability either. Some of them accept the entirety of classical mathematics but do not address the issue of applicability. Some of them try to develop subsystems of classical mathematics as philosophically more justifiable mathematics, for instance, constructivism, predicativism, and several programs for nominalizing a part of classical mathematics.[11] However, these subsystems all are committed to potential infinity, and applying them to strictly finite physical things in the universe above the Planck scale is still using infinite models to simulate strictly finite things. For instance, when we use a continuous function in intuitionistic mathematics to simulate the mass distribution of a fluid, the premises are again not literally true of those discrete particles in the fluid, and the logical puzzles of applicability remain the same.[12]

Note that I never assume that there is no real infinity in the physical world.[13] Most physicists today agree that current well-established physical theories accurately describe physical phenomena only above the Planck scale. As for what are below the Planck scale, physicists are still considering several possible theories, including discrete and non-4-dimensional space-time structures. This means that physicists believe that the successes of current theories above the Planck scale do not strictly imply the structure of space-time at the micro scale. That is, current theories are approximations only above the Planck scale. Even if physical space-time *is in fact* continuous, the validity of current theories above the Planck scale does not depend on this fact, and the logical puzzles of the applicability of infinite mathematical models in the current theories are still the same, and intuitively it is still reasonable to think that infinity is not strictly indispensable in the current theories. If someday physicists do confidently assert that space-time *is* continuous, then the logical puzzles of applicability may change or even disappear (in the case that there is no gap between the

[11] See [Burgess and Rosen, 1997] for a survey.

[12] See [Ye, forthcoming] for my general criticisms on current anti-realistic philosophies of mathematics.

[13] I would like to thank a referee for helping my clarification here.

mathematical model and physical space-time). This does not affect the puzzles of applicability in current theories. I will briefly discuss the philosophical value of studying applicability in current theories in Section 7.

5. A Plan for Explaining Applicability

The above analyses of the logical puzzle of applicability lead to a strategy for explaining applicability as follows. Using the notations in Section 3, we suspect that there are realistic premises (about concrete physical things) that are not explicitly included in Γ_r, but are implicitly implied by $\Gamma_m \cup \Gamma_b$. Suppose that Γ_r^* is the collection of realistic premises that we can excavate from $\Gamma_m \cup \Gamma_b$. Then, we suspect that, for any scientifically meaningful realistic conclusion φ drawn by scientists, we actually have $\Gamma_r \cup \Gamma_r^* \vdash \varphi$, where the deductions are valid in the naturalized sense. The truth of the realistic premises in Γ_r^* is implicitly accepted by scientists when they use $\Gamma_m \cup \Gamma_b$ to model real physical entities in that application. This, together with $\Gamma_r \cup \Gamma_r^* \vdash \varphi$, then implies that the realistic conclusion φ must also be literally true. That is, the explanation will go like this:

$$M_r \models \varphi, \text{ because } M_r \models \Gamma_r \cup \Gamma_r^* \text{ and } \Gamma_r \cup \Gamma_r^* \vdash \varphi.$$

To explain applicability, we must then excavate and identify such implicit realistic premises Γ_r^* and show that the original mathematical proof from $\Gamma_r \cup \Gamma_m \cup \Gamma_b$ to a scientifically meaningful conclusion φ can be transformed into a series of valid logical deductions from $\Gamma_r \cup \Gamma_r^*$ to φ.

Now, when infinite and continuous mathematical models are used to simulate finite and discrete phenomena, apparently we cannot translate mathematical premises into literally true realistic sentences about finite and discrete physical entities with the logical structures of those premises preserved. The mathematical premise stating the differentiability of a mass distribution function in a continuous model of fluid, for instance, cannot be so translated since a real fluid consists of discrete particles. Therefore, we cannot obtain Γ_r^* by translating mathematical premises and bridging postulations in $\Gamma_m \cup \Gamma_b$ into realistic assertions about physical entities directly. The case is even more complex when we use mathematical entities to simulate physical entities indirectly, for instance, using vectors in Hilbert spaces to simulate the states of quantum particles. Here, we appear to be using something alien to a physical system to encode information about the system. Accordingly, it is a genuine challenge to extract true realistic premises implicit in those infinite mathematical models and to transform proofs on mathematical models into logical deductions from realistic premises about discrete and finite real things alone.

To solve this problem, we use the following technical strategy. First, we define a logical framework called *strict finitism* for developing a kind of mathematics without infinity. Strict finitism is essentially a fragment of quantifier-free primitive recursive arithmetic (*i.e.*, PRA), with the accepted functions limited to elementary recursive functions.[14] Closed statements in strict finitism are reducible to the format $t = s$, where t and s are closed terms constructed from numerals and base elementary recursive functions by composition, bounded primitive recursion, finite sum, and finite product. We can interpret closed terms as programs (with fixed inputs) in computational devices (including brains). Then, $t = s$ says that two such programs will produce the same output. Some closed instances of an axiom in strict finitism can be interpreted as literally true statements about such programs in a finite computational device. Note that *not* all instances of an axiom schema can be so interpreted for a real computational device, because a real computational device has physical limits and cannot handle very large numerals properly. However, as long as the numerals involved are not too large and the computational device is functioning properly, an instance of an axiom can become literally true when interpreted into an assertion about that computational device.

Applying mathematics in strict finitism is essentially using a computational device (including a brain) to simulate other physical entities and their properties. We also have realistic premises, mathematical premises and axioms, and bridging postulations here. However, mathematical premises and the axioms of strict finitism are interpreted as statements about a computational device, and bridging postulations are interpreted as statements about how the computational device simulates those physical entities. These are all *realistic* statements. Therefore, an application is a series of logical deductions from *realistic* premises to a *realistic* conclusion.

So to explain the applicability of classical mathematics, we try to show that all scientifically valid applications of classical mathematics are in principle reducible to the applications of mathematics within strict finitism. Instead of translating the applications of classical mathematics into applications of strict finitism directly, our strategy is to develop *applied* mathematics within strict finitism. I will discuss how far this can be done in the next section. If successful, it will show that the language and logic of strict finitism are already sufficient for expressing current scientific theories about natural phenomena above the Planck scale and deriving scientific conclusions there. This means that we can *in principle* reformulate mathematical premises and bridging postulations in those applications as assertions about computational devices and their simulation relations with the physical entities to which we are applying mathematics. This should

[14] The language of strict finitism contains typed λ-calculus operators and allows the construction of terms representing typed functionals of natural numbers for encoding real numbers, functions of real numbers and so on. However, there are restrictions guaranteeing that all functionals constructed are elementary recursive in some appropriate sense. See [Ye, online-d; online-e] for more details.

not be very surprising. After all, from the point of view of a naturalistic observer, humans *are* actually using their brains, assisted by papers and pencils or computers, to simulate those physical entities when they apply *classical* mathematics to those physical entities. The only logical puzzle is that when they use classical mathematical concepts and thoughts that appear committed to infinity, the logic of how those concepts and thoughts simulate *finite* physical entities is not very clear. So our idea is that the convoluted logic in those abstract mathematical thoughts in classical mathematics can *in principle* be straightened to get logically simpler and more transparent (but much lengthier and more tedious) thoughts directly about finite computational devices and their simulation relations with physical entities.

These then constitute a logical explanation of the applicability of classical mathematics. In other words, in an application case like $\Gamma_r \cup \Gamma_m \cup \Gamma_b \vdash \varphi$, the implicit realistic premises Γ_r^* implied by $\Gamma_m \cup \Gamma_b$ mentioned above include the axioms of strict finitism, which state how brains or computers work as computational devices, and they also include other statements about how these computational devices simulate physical entities in the application. The fact that a scientifically valid application of classical mathematics is in principle reducible to an application of strict finitism (or the fact that strict finitism is in principle sufficient for the application) means that we then have $\Gamma_r \cup \Gamma_r^* \vdash \varphi$, whenever φ is a scientifically meaningful realistic conclusion. This is the explanation said to be wanted in the opening paragraph of this section.

To see how this explanation is naturalistic, consider the following type of explanation of physical phenomena. Suppose that we have a physical system, and suppose that, in a class A of state-transition processes for the system which we frequently observe, the end states always have a property T. Suppose that this regularity is not obvious from known physical laws, because the property T is not present at the initial states of those processes in A, and it is not preserved at the intermediate states in those processes either. Therefore, we have a puzzle. To resolve the puzzle, we analyze those processes and find that a state-transition process σ in A can be transformed into another state-transition process σ^* ending at the same end state. σ^* has an initial state with the property T, and it follows from known physical laws that the state transitions in σ^* preserve the property T. Therefore, it follows that the end state of σ^* will have T. This then demonstrates that the property T will present itself at the end state of the original process σ. Our strategy for explaining the applicability of classical mathematics is of the same kind, with the system being a scientist's brain (or her brain plus some pencil and papers or a computer), the processes being the inference processes in the brain in valid applications of *classical* mathematics, and the property T being the *naturalized* 'true' property for relevant *realistic* thoughts in the brain. The property T regularly presents itself at the end states of those applications of *classical* mathematics. However, it does not present itself at (or is not applicable to) the initial states, because the initial states involve abstract mathematical premises and bridging postulations (to which the naturalized 'true' property does not apply). Moreover, the property T is not preserved in the intermediate inference steps, because those steps may involve abstract mathematical thoughts as well. Our explanation says that such an inference process σ in a brain can *in principle* be transformed into another inference process σ^* in the brain. σ^* reaches the same realistic conclusion, but it starts from true realistic premises about the physical entities to which the brain is applying mathematics and about a computational device. Moreover, σ^* uses only valid logical inference rules (in the naturalized sense). Then, according to the known laws about the naturalized property 'true' for realistic thoughts and valid rules, the end state of σ^* will have the property 'true'. This then demonstrates that the property 'true' will present itself at the end state of the original inference process σ.

A few questions naturally arise about this explanation.[15] First, note that the hypothetical inference process σ^* and the computational device mentioned above do not really exist in the actual world. Is it legitimate for radical naturalism to refer to them? To see that we never go beyond nominalism and naturalism here, note that if a logician really wants to explain applicability in a specific instance of applying classical mathematics, she can study our technical work in developing applied mathematics within strict finitism and then go ahead and translate the application into an application of strict finitism. That will take some tedious logical and mathematical work, but accomplishing the work will actually realize the process σ^* above by her own brain and realize the virtual computational device (as a pencil and paper, or a computer). In other words, we offer a schema, and then everyone with sufficient patience can instantiate it in a concrete logically plain demonstration of why the conclusion drawn in a specific application of classical mathematics is true (in the naturalized sense). This follows the spirit of strict finitism, where we use concrete schemas to achieve generality but do not assume as irreducible primitives any abstract concepts such as 'an arbitrary intuitionistic proof' or 'computable functionals of finite types'. Similarly, the hypothetical inference process σ^* and computational device are certainly physically possible, but we do not really rely on the notion of possibility in any irreducible manner. Instead, we provide concrete instructions for creating the inference process σ^* and the computational device. Therefore, this is also different from the philosophical approaches by Chihara [1990] and Hellman [1989] which rely on the notion of modality in an essential and irreducible manner. Under

[15] I would like to thank two referees for raising some issues related to these questions.

radical naturalism, assuming an irreducible modality will cause difficulties. For instance, there is no naturalistic notion of truth for irreducible modal claims (*versus* the naturalized truth for realistic thoughts), and it is difficult to explain how *brains* as physical systems in the actual world can know irreducible modal truths. Only a naturalized modality is meaningful under naturalism, much like the naturalized truth.[16]

Second, is it legitimate for radical naturalism to resort to scientific laws that appear committed to abstract mathematical entities? If we really develop *all* scientific theories with strict finitism, then we actually refer only to concrete computational devices and physical entities. Then, our explanations of applicability do not refer to abstract mathematical entities and there is no circularity in this strategy of explaining applicability.

Third, in what sense does this explain applicability? Does it shift the *explanandum* from the original process σ to a different process σ^*? Admittedly, this is unlike an ordinary physics explanation where we explain an observed result by referring to its initial or boundary conditions and general laws. However, if successful, this does help resolve the logical puzzles of applicability, which are our real concern here. That is, from a given application of classical mathematics, this can demonstrate in plain logic how the conclusion logically follows from literally true premises. It also shows that assumptions about infinity and abstract mathematical entities are not among the logically minimum premises for deriving literal truths about real physical things. Moreover, there is certainly other logical research that one can do to clarify the logic of application. For instance, one can try to characterize a property 'approximately true of finite physical things' for abstract mathematical thoughts that appear committed to infinity and then examine how mathematical proofs in the applications preserve (or affect) this property.

6. The Conjecture of Finitism

Implementing this strategy relies on the following technical conjecture:

> ***The Conjecture of Finitism***: *Strict finitism is in principle sufficient for formulating current scientific theories about natural phenomena above the Planck scale and conducting proofs and calculations in those theories.*

Reasons supporting this conjecture include the fact that an impressive part of applied mathematics has been developed within strict finitism. The monograph [Ye, online-d] has developed the basics of calculus,

[16] See [Ye, 2009] for an effort to naturalize modality, and see [Ye, forthcoming] for more criticisms on the modal approaches in philosophy of mathematics.

metric space theory, complex analysis, Lebesgue integration theory, and the theory of unbounded linear operators on Hilbert spaces. These cover the essentials of *Constructive Analysis* [Bishop and Bridges, 1985] and the mathematics needed for the basics of classical quantum mechanics. Moreover, the general techniques in the monograph seem to show that applied mathematics within strict finitism can advance much further. Theorems in strict finitism have syntactical formats very similar to the corresponding theorems in classical mathematics. After a branch of applied mathematics is developed within strict finitism, we can rather straightforwardly translate physics textbooks written with that branch of applied classical mathematics into textbooks written with strict finitism. Physical theories will then have the same formal structures. Moreover, recall that we will need only finite precision in representing physical quantities above the Planck scale. Therefore, we have reason to believe that physical quantities and states in the actual applications can all be represented by functions available to strict finitism. Since the formal structure of a physics theory is preserved and real physical quantities can be represented, a physics theory formulated with strict finitism states the same *physical* facts and regularities as the original one formulated with classical mathematics. They are actually the same *physics* theory with different mathematical formalisms. Therefore, the development of applied mathematics within strict finitism supports the conjecture to some extent.[17]

There are also other intuitive reasons supporting the conjecture. For instance, the ratio between the linear cosmological scales and the Planck scale is less than 10^{100}. It seems that number-theoretic functions not essentially bounded by a few iterations of the power function do not have any chance to represent real physical quantities. This suggests that elementary recursive functions available to strict finitism may already be sufficient for encoding real numbers, functions of real numbers, and so on, for realistic applications. Moreover, since infinity, continuity, and so on, are only approximations to finite and discrete things above the Planck scale in the applications, intuitively we expect that infinity ought not to be strictly indispensable for deriving a physically meaningful conclusion. Otherwise, the conclusion may be physically unreliable, for the approximations are accurate only within some finite scope. This also hints that the applications are in principle reducible to the applications of strict finitism. For instance, the Jordan Curve Theorem in its original format may not be available to strict finitism. However, considering the fact that the space-time structure below the Planck scale is still unknown (and may be discrete or non-4-dimensional), we can expect that if the theorem is applicable in a real situation, what is really relevant for the application must be some approximate

[17] I would like to thank a referee for raising this question of how strict finitism is applied.

version of the theorem (*e.g.*, a discretized version) that does not take the continuity of space literally. Such a version is likely to be essentially finitistic.

Now, consider some possible counterexamples to the conjecture. Design a computer simulating the proofs in ZFC. We believe that the machine will never output $0 = 1$ as a theorem, which follows from our belief that ZFC is consistent. That belief about a concrete machine is perhaps not obtainable without entertaining the concepts and axioms in set theory. This appears to be a counterexample to the conjecture. However, from the naturalistic point of view, the belief in consistency is inductive in nature. In other words, after human brains practice entertaining concepts in set theory for a long time, and after obvious paradoxes are eliminated, the brains come to believe that no paradoxes will be derived in the future. This is essentially an inductive belief achieved by a brain based on reflections upon (*i.e.*, observing) its own activities. It should not be surprising that such a belief is not obtainable without entertaining those abstract mathematical concepts in brains, because it is just about what will happen in entertaining those concepts. As a case of scientific application, we take it that our inductive belief in the consistency of ZFC is among the premises for deriving our belief about that machine and that derivation is finitistic. Therefore, this is not a counterexample to the conjecture.

Similarly, recall that for a Π_1 arithmetic sentence φ in PRA, we have

$$\text{PRA} \vdash \text{Con}_{ZFC} \wedge \text{Pr}_{ZFC}(\#(\varphi)) \rightarrow \varphi,$$

where Con_{ZFC} states the consistency of ZFC, and Pr_{ZFC} is the proof predicate of ZFC, and $\#(j)$ is the Gödel number of the formula j. In strict finitism, believing the consistency of ZFC similarly implies believing the (quantifier-free) arithmetic formulas of strict finitism derivable from ZFC.[18] As for an arbitrary first-order arithmetic formula, if it is to be meaningful for real things in this universe, from the Planck scale to the cosmological scales whose ratio is less than 10^{100}, we can perhaps expect that all its quantifiers are actually bounded by elementary recursive functions. Then, it is reduced to an arithmetic formula of strict finitism. This means that beliefs about strictly finite concrete things in this universe obtained by applying first-order arithmetic formulas provable from ZFC are accountable as finitistic consequences of the inductive belief in the consistency of ZFC. Therefore, we will not get any counterexample to the conjecture by applying ZFC in this manner.

These reasons are still far from conclusive. More work has to be done in developing applied mathematics within strict finitism, as well as in

[18] The arithmetic formulas of strict finitism are those constructed from $=$, $\neg$, $\rightarrow$, variables, symbols for elementary recursive functions, and bounded quantifiers. See [Ye, online-d] for the details.

analyzing what could be a counterexample to the conjecture, in order to support the conjecture better. However, based on the reasons we already have, a positive answer to the conjecture seems *plausible*. So this plan for explaining applicability seems a workable plan.

Finally, remember that I take a completely scientific attitude here. I am *not* trying to look for an *a priori* argument that the conjecture of finitism must be true. Even if we end up with a negative answer, it will also be a valuable thing to know where exactly infinity is strictly logically indispensable for an application to finite things in this physical world and how that can happen. In that case, we may have to admit that the applicability of infinity to finite things in some cases simply defies any plain logical explanation (even if we accept realism). Moreover, recall that I never assume that there is no infinity in the physical world. That question should be left for physicists to decide. The real point is that current well-established theories are about only a finite part of the physical world.

7. A Few Final Remarks

This strategy for explaining applicability is similar to Hartry Field's strategy for demonstrating the conservativeness of classical mathematics over a nominalistic physics-cum-mathematics theory [Field, 1980; 1998]. However, Field assumes infinity and continuity of space-time. We do not want our philosophical and logical explanation of the applicability of mathematics to areas such as economics to rely on a physical assumption about space-time, not to mention its being an assumption that is still undecided today. More importantly, Field's strategy cannot resolve the logical puzzles of applicability discussed here. This strategy is also similar to Hilbert's program, which tries to get a once-and-for-all proof of the conservativeness of classical mathematics over finitism [Hilbert, 1925]. We know that this cannot succeed because of Gödel's Incompleteness Theorems, and we know that classical mathematics is not conservative over finitism. This approach intends to show that a finitistic mathematical system is already in principle sufficient for current scientific applications and this implies that the part of classical mathematics that is actually applied in the sciences *is* conservative over finitism.[19]

I have no intention of suggesting adopting strict finitism in place of classical mathematics for scientific applications. This is meant to be a scientific study of the successes of applying *classical* mathematics as natural phenomena, aiming to resolve some logical puzzles there. The reductions to strict finitism never happen in scientists' brains, and certainly the brain

[19] See also [Ye, 2000, Appendix C] for a comparison between reverse mathematics by Friedman and Simpson [Simpson, 1988] and the system in [Ye, 2000], which is a little stronger than the latest system in [Ye, online-d].

processes through classical mathematics are much simpler than the brain processes through strict finitism. It requires great ingenuity and is a superb scientific achievement to have discovered simple and effective thought processes for simulating very complex things in the real world sufficiently accurately. A logician's job is to resolve the logical puzzles in their ingenious inventions, and strict finitism is invented as a logical analytical tool to assist in that purpose. It is certainly not meant to replace scientists' ingenious inventions.

Moreover, I do not mean that strict finitism is the only true mathematics or the foundation of meaningful mathematics. Firstly, the axioms of strict finitism are abstract thoughts in brains and the naturalized property 'true' does not apply to them either. Secondly, some of these axioms can be interpreted into true realistic thoughts about concrete computational devices, but many of them have no such chance because there are no sufficiently large concrete computational devices in the universe. Finally, for a true naturalist, the quest for an absolutely certain foundation of knowledge is pointless. All knowledge in a brain comes from the innate cognitive architecture of the brain determined by genes and the physical interactions between the brain and its environment. The idea of absolutely certain knowledge seems to presuppose an absolute, non-material, and transcendental cognitive subject, which is alien to naturalism.[20]

Finally, I do not mean that the validity of nominalism depends on the status of the Conjecture of Finitism, and it is not the goal of this paper to defend nominalism. If space-time *is* continuous, certainly some infinite mathematics is needed for physics, but that mathematics may have a nominalistic interpretation as a theory about physical structures. Therefore, this possibility might not invalidate nominalism, although it would show that nominalism is independent of the conjecture. I believe that nominalism follows from a coherent understanding of naturalism. (See [Ye, online-b] for an argument.) So, under naturalism, applicability becomes a logical problem. However, if we can indeed explain the applicability of classical mathematics in *current well-established scientific theories* about natural phenomena above the Planck scale by eliminating infinity, the result *does* support nominalism by discrediting the indispensability argument for realism. The idea of the argument is that truths about things essentially beyond the physical world (*e.g.*, not structurally isomorphic to any physical things) are indispensable for the scientific descriptions of the physical world. If our strategy for explaining applicability is successful, then this idea is wrong, at least for current well-established theories and for strict indispensability (not pragmatic indispensability). The success of our strategy would

[20] A brain can certainly reorganize its knowledge base and make it more reliable and efficient, but reducing to strict finitism is extremely inefficient and the gain in reliability insignificant.

suggest that there is a misconception in the indispensability argument regarding why and how mathematics is applicable. That is, mathematics is applicable not because it is a collection of literal truths about mathematical entities, but because it is a coherent fiction that can provide *simple but sufficiently accurate* fictional models for simulating real things in the physical world. So the pragmatic indispensability of classical mathematics for science would also be accountable under nominalism. It consists of the pragmatic values of fictions in representing relevant aspects of the physical world sufficiently accurately, efficiently, and with prediction and explanatory powers.[21]

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[21] This fictionalist account of mathematics has been endorsed by many philosophers, but I believe that mere fictionalism is inadequate and that refuting strict indispensability is necessary. See [Ye, forthcoming].

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