A STRICTLY FINITISTIC SYSTEM FOR APPLIED
MATHEMATICS

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Abstract. This paper reports the technical work in the monograph

*Strict Finitism and the Logic of Mathematical Applications* (available
online). The monograph proposes a strictly finitistic system and then
showes that some significant applied mathematics, including the basics
of unbounded linear operators on Hilbert space, can be developed within
that system. Potential philosophical implications or applications of the
work are briefly discussed.

1. Introduction

This paper reports the technical work in the monograph *Strict Finitism
and the Logic of Mathematical Applications* (Ye [12]). The monograph
develops a system of finitistic mathematics, called **Strict Finitism**. It is
essentially a fragment of the quantifier-free primitive recursive arithmetic
(PRA) with the accepted functions restricted to elementary recursive func-
tions. The monograph shows that some significant applied classical math-
ematics can be developed within strict finitism. So far this includes the

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basics of calculus, metric space theory, complex analysis, Lebesgue integration theory, and the theory of unbounded linear operators on Hilbert spaces. To allow encoding real numbers, functions of real numbers and so on, we allow the language of strict finitism to include typed variables, λ-abstractions, and functional applications, but we make some restrictions carefully so that this is not an essential extension. All numerical functions constructed in strict finitism are still elementary recursive functions only.

Some philosophers argued that PRA correctly represents the spirit of Hilbert’s finitism (e.g. Tait [7]), and doubts about that claim usually point to the direction that finitism may be stronger than PRA. As a fragment of PRA, strict finitism is perhaps more obviously strictly finitistic. One of the goals for developing strict finitism is to look into what exactly is the logical strength of the minimum system of mathematics sufficient for scientific applications. The reason for restricting to elementary recursive functions is to recognize the fact that, in scientific applications, elementary recursive functions seem to be sufficient for encoding real numbers, functions of real numbers and other entities or structures. I will come back to this point in the last section of this paper.

The development of mathematics in strict finitism adopts its main ideas from Bishop’s constructive mathematics (Bishop and Bridges [5]). The logical basis of Bishop’s constructive mathematics is stronger than finitism. Part of our work consists in revising Bishop and Bridges’ definitions and
proves to fit into our more restrictive framework. This paper is a short report of the work. Details omitted here can be found in the monograph (Ye [12]).

This work is a continuation of some previous work (Ye [9], [10]). Ye [10] works within the framework of Bishop’s constructive mathematics. Ye [9] works within a more restrictive framework that is essentially equivalent to PRA. This work finally reduces the basis to strict finitism, namely, quantifier-free elementary recursive arithmetic. For that, the theory of (Lebesgue) integration has to be completely revised. Otherwise, reducing the work in Ye [9] to the strict finitism here mostly consists in technical improvements.

The fact that some significant applied classical mathematics can be developed within strict finitism may have some philosophical implications, or one may try to utilize such technical results for defending some philosophical positions, but here I like to caution that I have no intention to suggest that strict finitism is the right mathematics for applications, or that strict finitism is the only meaningful mathematics, or even that strict finitism is a better mathematics than classical mathematics in any interesting sense. I do have my own philosophical positions regarding issues in philosophy of mathematics, but they are different from what are commonly named ‘finitism’ in the literature (Ye [11]). My own intended use for strict finitism is as an assistant logical tool for explaining the applicability of classical mathematics in the sciences. I will briefly explain this use of strict finitism in the
last section of this paper. However, it seems worthwhile to explore what the
minimum mathematical system for scientific applications is and explore if
strict finitism is in principle sufficient for scientific applications, irrespective
of any philosophical issues. In the last section of this paper, I will discuss
some intuitive reasons for thinking that strict finitism may be in principle
sufficient for scientific applications.

2. The Formal System $\mathbf{SF}$

The formal system $\mathbf{SF}$ is the base system for strict finitism. The language
of $\mathbf{SF}$ is the language of typed $\lambda$-calculus (Barendregt [2]), plus constants
for 0 and the base elementary recursive functions, and plus operators for
bounded primitive recursion, finite sum, finite product and definition by
cases. They are summarized as follows:

**Types:** $\sigma$ is a type (the base or numerical type), and if $\sigma_1, \ldots, \sigma_n, \sigma$ are
types, then $(\sigma_1, \ldots, \sigma_n \rightarrow \sigma)$ is a type.

**Variables:** For each type $\sigma$, there are variables $x_1^\sigma, x_2^\sigma, \ldots$ of the type.

**Constants:** 0, $S$, $\cdot$, $\cdot$, $\text{pow}$, $I_<$ for the base elementary recursive func-
tions.

**Terms:** Terms include variables, constants, functional applications $Ap\,(t, s_1,\ldots, s_n)$
(also denoted as $t\,(s_1,\ldots, s_n)$), and $\lambda$-abstractions $\lambda x_1^{\sigma_1} \ldots x_n^{\sigma_n}.t$ as in typed $\lambda$-
calculus, plus the following:
(1) If \( t \) is a term of the type \( o \), and \( t_1, t_2 \) are terms of the type \( \sigma \), then \( J(t, t_1, t_2) \) is a term of the type \( \sigma \). \( J(t, t_1, t_2) \) is to mean definition by cases, that is, it is \( t_1 \) or \( t_2 \) when \( t = 0 \) or \( t > 0 \) respectively.

(2) If \( t[i, j], r, s \) are terms of the type \( o \), \( b \) is a term of the type \( (o \rightarrow o) \), \( i, j \) are variables of the type \( o \), and \( i, j \) are not free in \( b, r, s \), then \( Re_{ij}(s, r, b, t \{i, j\}) \) is a term of the type \( o \). \( Re_{ij}(s, r, b, t \{i, j\}) \) is to mean bounded primitive recursion restricted to numerical functions, where \( s \) is the number of recursive steps, and \( r \) is the initial value, and \( b \) gives the bound, and \( t \{i, j\} \) gives the recursion scheme.

(3) If \( t[i] \), \( r \) are terms of the type \( o \), then \( \sum_{i \leq r} t \{i\}, \prod_{i \leq r} t \{i\} \) are terms of the type \( o \).

Formulas: Formulas include atomic formulas \( t = s \) for \( t, s \) of the type \( o \), and boolean combinations of these.

A term \( S...S0 \) is a numeral. We define \( s < t \equiv_{df} I_<(s, t) = 0 \), \( s \leq t \equiv_{df} s < t \lor s = t \). We will write \( pow(s, t) \) as \( s^t \). Note that while all elementary recursive functions can be constructed from \( S \) and \( x^y \) by composition and bounded primitive recursion in classical mathematics, we need \(+, \cdot, \text{ and } I_<\) as primitives in order to state the primitive recursive equations for \( x^y \) and to express the ‘bound’ in a bounded primitive recursion in \( SF \). Note that \( t = s \) is a formula only if \( t, s \) are numerical terms. There are no equalities between higher-type terms in the language of \( SF \). Also note that there are no quantifiers in the language of \( SF \). Generality has to be achieved by using
free variables or by making schematic assertions about arbitrary terms or formulas of some format.

**Axioms:** The axioms of SF include the axioms of classical propositional logic, the axioms characterizing the equalities \( t = s \), the arithmetic axioms characterizing \( S \) as in Peano arithmetic, the primitive recursive definitions for \( +, \cdot, pow, I_\prec \), finite sum \( \sum_{i \leq r} t [i] \), and finite product \( \prod_{i \leq r} t [i] \), the axioms in typed \( \lambda \)-calculus for functional applications and \( \lambda \)-abstractions, and the following additional axioms:

1. **Selection Axioms** (Note: \( s \{t\} \) indicates an occurrence of a subterm \( t \) in the term \( s \).):

   \[
   s \{ J (0, t_1, t_2) \} = s \{ t_1 \}, \quad s \{ J (St, t_1, t_2) \} = s \{ t_2 \},
   \]

   \[
   s \{ Ap (J (t, t_1, t_2), s) \} = s \{ J (t, Ap (t_1, s), Ap (t_2, s)) \},
   \]

   \[
   s \{ \lambda x . J (t, t_1, t_2) \} = s \{ J (t, \lambda x . t_1, \lambda x . t_2) \};
   \]

2. **Recursion Axioms**:

   \[
   Re ij (0, r, b, t [i, j]) = J (I_\prec (r, b (0)), r, b (0)),
   \]

   \[
   Re ij (Ss, r, b, t [i, j]) = J (I_\prec (t', b (Ss)), t', b (Ss)),
   \]

   where \( t' \equiv t [s, Re ij (s, r, b, t [i, j])] \);

**Rules:**

1. **Modus Ponens:** \( \varphi \rightarrow \psi, \varphi \implies \psi; \)

2. **Induction:** \( \varphi [0], \varphi [n] \rightarrow \varphi [Sn] \implies \varphi [t]. \)
Note that axioms in (1) are stated in schematic formats, because we do not have equalities between the terms of higher-types and we cannot state the Selection Axioms as \( J(0, t_1, t_2) = t_1, J(St, t_1, t_2) = t_2 \). The same applies to the axioms for functional application and \( \lambda \)-abstraction.

The concept of Normal Form in classical typed \( \lambda \)-calculus applies here as well, and we also have a Normal Form Theorem. Then, it can be proved that if \( t[m_1, ..., m_l] \) is a numerical term in normal form and its free variables are all among the numerical variables \( m_1, ..., m_l \), then \( t[m_1, ..., m_l] \) contains no occurrences of \( \lambda \). This means that a numerical term \( t[m_1, ..., m_l] \) with only numerical free variables represents an elementary recursive function.

\( SF \) is closely related to the system \( \mathbf{T}_0 \) in Avigad and Feferman [1], or the system \( \mathbf{T}_0 \) defined in Troelstra [8] with recursion operators restricted to numerical functions. \( SF \) is a proper subsystem of these systems because it admits only bounded primitive recursions, and thus it can represent elementary recursive functions only, not all primitive recursive functions.

Closed terms in \( SF \) can be interpreted as computer programs (without inputs) producing numerals. In particular, a closed numerical term in normal form is a composition of numerals, the base elementary recursive functions \( S, +, \cdot, \text{pow} \), and \( I_\prec \), bounded primitive recursion, finite sum, and finite product. It can be interpreted as a program producing a concrete numeral output when executed (according to the axioms). An arbitrary closed numerical term can also be interpreted as a program, since it can be transformed into a normal term. A closed atomic formula \( t = s \) can
then be interpreted as an assertion about two such programs, saying that they output the same numeral. This is one of the potential strictly finitistic interpretations of $\text{SF}$.

Note that when interpreted as assertions about a concrete computer realizing the programs in the real world, not all closed instances of the axioms are literally true of a concrete computer, because we have to consider the physical limitation of a concrete computer. For instance, the function symbol $S$ is interpreted as the computer operation of adding 1 to a numeral. Since to some point this will cause overflow in a real computer, some instances of the axiom $St = Sr \rightarrow t = r$ may not be literally true when so interpreted. However, as long as the numerals involved are not too large, an axiom instance can indeed be interpreted as a true assertion about programs in a concrete computer, and more instances of the axioms can be so interpreted when we consider physically possible computing devices. Therefore, we treat the sentences of $\text{SF}$ as uninterpreted formal sentences, which may or may not have a chance to be interpreted into statements about real things. I will not discuss here the philosophical question if the axioms of $\text{SF}$ are true in themselves, but see Ye [11].

3. Doing Mathematics in Strict Finitism

Some basic arithmetic theorems can be directly stated and proved in $\text{SF}$. Moreover, bounded quantifiers and bounded minimalization can be defined in $\text{SF}$, and with these, we can develop encodings for finite sequences of
numerals and we can then define other elementary recursive functions and predicates.

To express more advanced mathematics in strict finitism, we need some assistant notations. We consider developing mathematics in strict finitism to be constructing terms of $\text{SF}$ and proving that the constructed terms satisfy some conditions, where the conditions are expressed as quantifier-free formulas in $\text{SF}$. This is much like a computer programmer’s job, namely, designing programs and demonstrating that the programs designed meet given specifications. These programs are then the resources for simulating other things in the world in applications. We will introduce some notations to allow stating, in a simplified manner, what terms of $\text{SF}$ have been constructed and which conditions expressed as quantifier-free formulas in $\text{SF}$ are verified. In particular, we want the simplified statements to look similar to statements in classical mathematics (actually, in Bishop’s constructive mathematics).

First, we use the notation $(x, y, p$ denote sequences of distinct variables) 

$\exists x\forall y \varphi [x, y, p]$

(FinC)

to mean that we have constructed a sequence of terms $t$ of appropriate types and prove that

$\text{SF} \vdash \varphi [t, y, p]$,
where \( \varphi \) is a quantifier-free formula in \( \text{SF} \), and \( t \) may contain variables in \( p \) but not variables in \( y \). This will be called a **claim in strict finitism**. Variables \( p \) are free variables as parameters in the claim. A **proof of the claim** in strict finitism consists of the required terms \( t \) and a proof of \( \varphi[t, y, p] \) in \( \text{SF} \). The constructed terms \( t \) will be called **witnesses** for the claim. Therefore, doing mathematics in strict finitism means proving claims in strict finitism.

The existential quantifier here only means that relevant terms have been constructed, and the universal quantifier is only for indicating free variables independent of the constructed terms in the condition to be verified within \( \text{SF} \). Both quantifiers are not understood in the classical or even the intuitionistic sense. The symbols \( \exists \) and \( \forall \) will occur **only in such contexts** and other ways of nesting them are meaningless. We use the existential quantifier because we do not want to mention the details of those constructed terms in the claim. Our interest is only to communicate the fact that they have been constructed. A proof of the claim must explicitly contain the terms required. We will accept informal arguments demonstrating that such terms can be constructed, but the informal arguments must allow extracting such terms from the arguments, and this ‘allow extracting’ must itself be understood in the strictly finitistic sense (e.g. by an elementary recursive function). (A programmer cannot sell a program by theoretically proving its existence. He or she must sell a finished program, or at least a design from which a program can be rather straightforwardly extracted.)
Then, we introduce some new and defined logical constants $\neg^*$, $\lor^*$, $\land^*$, $\forall^*$, and $\exists^*$ in our semi-formal language, to allow expressing claims like $(\text{FinC})$ in strict finitism in more readable formats and to allow more familiar informal arguments for proving claims like $(\text{FinC})$ in strict finitism. These logical constants are explicitly defined. They may not be equivalent to their corresponding classical or intuitionistic logical constants. More specifically, suppose that $\varphi \equiv \exists x \forall y \varphi_1[x, y]$ and $\psi \equiv \exists u \forall v \psi_1[u, v]$ are claims in strict finitism. (We suppress the parameters here.) Define

1. $(\varphi \land^* \psi) \equiv_{df} \exists x u \forall y v (\varphi_1 \land \psi_1)$;
2. $(\varphi \lor^* \psi) \equiv_{df} (\varphi_1 \lor \psi_1)$ if $x, y, u, v$ are all empty; otherwise,

\[
(\varphi \lor^* \psi) \equiv_{df} \exists n x u \forall y v ((n = 0 \land \varphi_1) \lor (n \neq 0 \land \psi_1));
\]

3. $(\exists^* z \varphi) \equiv_{df} \exists z x \forall y \varphi_1$ if $z$ does not occur in $x, y$, otherwise $(\exists^* z \varphi) \equiv_{df} \varphi$;
4. $(\forall^* z \varphi) \equiv_{df} \exists X z \forall y \varphi_1[X(z), y]$ if $z$ does not occur in $x, y$, otherwise $(\forall^* z \varphi) \equiv_{df} \varphi$;

(5) $(\varphi \rightarrow^* \psi) \equiv_{df} \exists U Y x v (\varphi_1[x, Y(x, v)] \rightarrow \psi_1[U(x), v])$;
(6) $(\neg^* \varphi) \equiv_{df} (\varphi \rightarrow^* S 0 = 0) \equiv \exists Y x (\neg \varphi_1[x, Y(x)])$;
(7) $(\varphi \leftrightarrow^* \psi) \equiv_{df} (\varphi \rightarrow^* \psi) \land^* (\psi \rightarrow^* \varphi)$.

We can use these defined logical constants to construct new claims in strict finitism, as in the language of quantified type theory. For a formula $\varphi$ constructed using these defined logical constants from claims in the format
(FinC), after these defined logical constants are eliminated, it will eventually reduce to a claim in the format (FinC) again.

These definitions are essentially Gödel’s *Dialectica* interpretation of the intuitionistic logic. We can prove that all defined logical constants $\rightarrow^*$, $\lor^*$, $\land^*$, $\rightarrow^*, \leftrightarrow^*$, $\exists^*$, and $\forall^*$ follow the intuitionistic logical laws, including the axiom of choice. For instance, for the logical axiom $\psi \rightarrow^* (\varphi \rightarrow^* \psi)$, this means that when the defined constant $\rightarrow^*$ is eliminated, we get a claim in the format (FinC), and the required witnesses for the claim can be automatically constructed. Therefore, we can use intuitionistic logic for deriving a claim of strict finitism stated using these defined logical constants. Then, witnesses for the derived claim can be automatically extracted from the proof, and we get a proof of the claim in the format (FinC) above in strict finitism.

The definition of $(\varphi \rightarrow^* \psi)$ above is Bishop’s numerical implication in Bishop [4]. Intuitively, it means that to claim that $\exists u \forall v \psi_1 [u,v]$ follows from the assumption $\exists x \forall y \varphi_1 [x,y]$ in strict finitism, one must take an arbitrary $x$ and derive $\exists u \forall v \psi_1 [u,v]$ from $\forall y \varphi_1 [x,y]$, which means that one must construct a term $U$ that operates on arbitrary $x$ and derive $\psi_1 [U (x) , v]$ from $\forall y \varphi_1 [x,y]$, which in turn means that one must construct $Y$, operating on $x,v$, and derive $\psi_1 [U (x) , v]$ from $\varphi_1 [x,Y (x,v)]$. This is at least as strong as the intuitionistic implication, in the sense that if $\varphi$ and $\psi$ have the formats above and $\varphi \rightarrow^*$ is provable in strict finitism, then ‘if $\varphi$, then $\psi$‘ is provable in intuitionism. A natural question is: is this numerical interpretation of implication too strong? In particular, in deriving
$\psi_1[U(x), v]$, it allows using an instance $\varphi_1[x, Y(x, v)]$ only, not the full universal claim $\forall y \varphi_1[x, y]$. It can be proved that, for deriving $\psi_1[U(x), v]$ in strict finitism, one can actually use finitely many instances of $\forall y \varphi_1[x, y]$, that is, $\forall n \leq M(x, v) \varphi_1[x, Y'(n, x, v)]$ for some term $M(x, v)$, where the number of instances can depend on $x, v$. Still, it does not allow operating on ‘an arbitrary proof ’ of $\forall y \varphi_1[x, y]$ as it is allowed by intuitionism, nor does it allow depending on infinitely many instances of $\forall y \varphi_1[x, y]$. This, we believe, captures the finitistic numerical content of implication. It means that, for finitistic implication, the universal quantifier in the antecedent should not be taken too literally. A proof of the implication must derive the consequent $\psi_1[U(x), v]$ from finitely many explicitly constructed instances of the antecedent $\forall y \varphi_1[x, y]$. This reflects the fact that in using infinite and continuous models to simulate finite and discrete things in applications, we do not take our premises too literally. See Ye [12] for more on this.

These defined logical constants allow us to state claims in strict finitism in a manner that looks very close to statements in classical mathematics. It is easy to see that starred logical constants and non-starred logical constants are compatible when both are meaningful in a context. Therefore, we can omit the stars on those logical constants. Then, when developing mathematics in strict finitism, we translate a theorem in classical mathematics into a claim in strict finitism, with the logical constants in classical mathematics translated into $\neg^*, \lor^*, \land^*, \rightarrow^*, \leftrightarrow^*, \exists^*$, and $\forall^*$. (We frequently state such claims in natural language, instead of using symbolic formulas.)
Every mathematical theorem that we can prove in strict finitism is then eventually a claim in the format (FinC) above, stating that some terms in \( SF \) have been constructed and some conditions involving those terms have been verified within \( SF \). These translations from the theorems of classical mathematics to claims in strict finitism assign numerical content to the classical theorems. Proving a theorem in strict finitism means constructing the relevant terms realizing the numerical content, that is, terms satisfying some relevant conditions expressed by the (quantifier-free) formulas of \( SF \).

Since the intuitionistic logical laws hold for the defined logical constants \( \neg^*, \lor^*, \land^*, \rightarrow^*, \leftarrow^*, \exists^* \), and \( \forall^* \), proving claims in strict finitism will be very close to proving theorems in Bishop’s constructive mathematics (Bishop and Bridges [5]). The only essential difference is that strict finitism allows bounded primitive recursive constructions and quantifier-free inductions only, and it recognizes elementary recursive functions only. Therefore, our work in developing mathematics within strict finitism frequently consists in unraveling the recursive constructions and inductions in constructive mathematics and reducing them into recursive constructions and inductions available to strict finitism.

4. Sets and Functions in Strict Finitism

Sets in strict finitism are actually formulas considered as conditions for classifying terms; functions are terms that apply to terms satisfying some conditions and produce other terms satisfying some other conditions. Sets
and functions provide a convenient way to express conditional constructions. That is, assuming that a given term satisfying some condition has been constructed, a function (as a term) will operate on it and produce another term satisfying another condition. Sets and functions together allow us to express sophisticated conditional constructions of terms and express conditions about terms in more readable and familiar formats. The idea of sets and functions here are from Bishop and Bridges [5], with some necessary revisions to fit into our more restrictive framework.

Given a pair \( A \equiv \langle \varphi [a] , \psi [a, b] \rangle \) of formulas in the extended language, where \( a, b \) are variables of an arbitrary type \( \sigma \), ‘\( A \) is a set of the type \( \sigma \)’ as a claim in strict finitism is the conjunction of the following two formulas (Note: we write \( \varphi [a] \) as \( a \in A \), and write \( \psi [a, b] \) as \( a =_A b \), called the membership condition and equality condition for the set respectively.):

\[
\begin{align*}
(1) & \forall a, b, c \left( a \in A \land b \in A \land c \in A \rightarrow \\
& \left( a =_A a \land (a =_A b \rightarrow b =_A a) \land (a =_A b \land b =_A c \rightarrow a =_A c) \right) \right), \\
(2) & \forall a, b (a \simeq b \land a \in A \rightarrow b \in A \land a =_A b).
\end{align*}
\]

Here, \( a \simeq b \) means the extensional equality, that is,

\[
a \simeq b \equiv \forall x_1 \ldots x_n \left( a(x_1) \ldots (x_n) = b(x_1) \ldots (x_n) \right),
\]

where \( x_1, \ldots, x_n \) are sequences of variables of appropriate types so that \( a(x_1) \ldots (x_n) \) becomes a numerical term. Therefore, the equality \( =_A \) is a more coarse-grained equivalence relation than the extensional equality. If \( (a \in A) \) is \( \exists x \forall y \varphi_0 [a, x, y] \) for \( \varphi_0 \) a formula of \( \text{SF} \), and \( x \) are of the types \( \rho_1, \ldots, \rho_m \),
we call \((\sigma, \rho_1, \ldots, \rho_m)\) the signature of the set \(A\) and call \(\rho_1, \ldots, \rho_m\) the witness types of \(A\), and we use \(a \in_x A\) to denote the formula \(\forall y \varphi_0 [a, x, y]\), read as ‘\(a\) belongs to \(A\) with \(x\) as the witnesses’.

For instance, we assume a fixed encoding of rational numbers and then we can define the set \(\mathbb{Q}\) of rational numbers. Then, the set of real numbers, \(\mathbb{R}\), is a set of the type \(\sigma = (o \rightarrow o)\) and the signature \((\sigma)\) (see Bishop and Bridges [5], p. 18):

\[
x \in \mathbb{R} \equiv_{df} \forall m, n > 0 (x(n) \in \mathbb{Q} \land |x(m) - x(n)| \leq 1/m + 1/n),
\]
\[
x =_{\mathbb{R}} y \equiv_{df} \forall n > 0 (|x(n) - y(n)| \leq 2/n).
\]

Therefore, we essentially take elementary recursive Cauchy sequences of rational numbers as real numbers. We can define \(x < y \equiv_{df} (\exists n > 0) (x(n) < y(n) - 2/n)\).

Then, similarly, the set \(\mathbb{R}^+\) of positive real numbers has the signature \(((o \rightarrow o), o)\),

\[
x \in \mathbb{R}^+ \equiv_{df} x \in \mathbb{R} \land x > 0;
\]

Note that \(x > 0\) requires a witness. Therefore, the membership condition \(x \in \mathbb{R}^+\) requires a witness.

Unions, intersections, and other operations on sets of the same type can be defined; the inclusion and extensional equality relation between sets can be similarly defined.
A function is a term that applies to terms belonging to the domain set of the function together with their witnesses and results in a term belonging to the range set. Suppose that $A$ and $B$ are sets of the signatures $(\sigma_0, \sigma_1, \ldots, \sigma_n)$ and $(\rho_0, \rho_1, \ldots, \rho_m)$ respectively, and $f$ is a term of the type $(\sigma_0, \sigma_1, \ldots, \sigma_n \to \rho_0)$. ‘$f$ is a function from $A$ to $B$’, or ‘$f : A \to B$’, is the claim

$$\forall x_0 x_1 (x_0 \in_{x_1} A \to f(x_0, x_1) \in B) \land$$

$$\forall x_0 x_1 y_0 y_1 (x_0 \in_{x_1} A \land y_0 \in_{y_1} A \land x_0 =_A y_0 \to f(x_0, x_1) =_B f(y_0, y_1)).$$

From $f : A \to B$ it follows that $x_0 \in_{x_1} A \land x_0 \in_{x_2} A \to f(x_0, x_1) =_B f(x_0, x_2)$. So we frequently simply write $f(x_0)$ instead of $f(x_0, x_1)$, as equal members of a set can be treated as the same in most contexts. Similarly, sometimes we use notations like $\forall x_0 \in A (\ldots f(x_0) \ldots)$, while literally it should be $\forall x_0 x_1 (x_0 \in_{x_1} A \to \ldots f(x_0, x_1) \ldots)$. Such simplified notations are more readable, and contexts can always determine how to complete them, as long as we always keep in mind that a function operates on elements in its domain as well as the witnesses for elements belonging to its domain.

For example, the function $x^{-1}$ on the set $\mathbb{R}^+$ operates on an arbitrary sequence $x$ of rational numbers and an arbitrary natural number $m$ as the potential witness for $x \in \mathbb{R}^+$, that is, a number such that $x(m) > 2/m$. It’s easy to see that for $x \in \mathbb{R}$ and $m$ such that $x(m) > 2/m$, we can find $N, M$, such that for all $n, k \geq N$, we have $x(nM) > 0$ and

$$|x(nM)^{-1} - x(kM)^{-1}| \leq 1/n + 1/k.$$
that \( t(x,m)(k) = x(NM)^{-1} \) for \( k \leq N \) and \( t(x,m)(k) = x(kM)^{-1} \) for \( k > N \). Then, it is easy to see that \( t(x,m) \in \mathbb{R} \) if \( x \in m \mathbb{R}^+ \). \( t \) is then the inverse function \(-1\) on the set \( \mathbb{R}^+ \). It applies to \( x \) and a witness \( m \) such that \( x \in m \mathbb{R}^+ \), and produces \( t(x,m) \in \mathbb{R} \).

We can also define sets of functions. Given sets \( A \) and \( B \), the set of functions from \( A \) to \( B \), denoted as \( F(A,B) \), is defined as follows:

\[
f \in F(A,B) \equiv_{df} f : A \to B,
\]

\[
f =_{F(A,B)} g \equiv_{df} \forall x_0 x_1 (x_0 \in A \to f(x_0, x_1) =_B g(x_0, x_1)).
\]

This equality condition, which can be expressed as \( \forall x \in A (f(x) = g(x)) \) in a simplified manner, is called extensional equality for functions. We agree that when defining sets of functions, for instance, the set of continuous functions on \( \mathbb{R} \) and so on, the extensional equality for functions is always assumed unless otherwise stated.

Note that ‘\( A \) is a set’, ‘\( f : A \to B \)’ and so on are only convenient ways for stating claims in strict finitism. Apparent references to ‘set’, ‘function’ can be eliminated if we spell out the claims. For instance,

\[
(f : \mathbb{R}^+ \to \mathbb{R}) \land \forall x \in \mathbb{R}^+ (x \cdot f(x) = 1)
\]

gives the condition for a term \( f \) to be the function \( x^{-1} \) on \( \mathbb{R}^+ \). Spelling it out, we will get a very complex statement using the symbols \( \neg^*, \lor^*, \land^*, \to^*, \exists^*, \forall^* \), as well as the symbols \( \neg, \lor, \land, \to \) in \( SF \). Then, after those defined logical constants \( \neg^*, \lor^*, \land^*, \to^*, \exists^*, \forall^* \) are eliminated, it
eventually becomes a claim in strict finitism in the format (FinC), stating that some terms of SF can be constructed, together with \( f \), to satisfy some condition expressed as a quantifier-free formula in SF. Introducing ‘set’ and ‘function’ and so on greatly simplifies our notations. On the other side, this also means that we cannot quantify over sets. We can only make schematic assertions involving sets defined by arbitrary formulas in some format.

5. Applied Mathematics in Strict Finitism

5.1. Calculus. The set \( \mathbb{R} \) of real numbers has been defined above. A sequence \((a_n)\) of real numbers is a term \( a_n \) of the type \((o \to o)\) with a free variable \( n \) of the type \( o \). A sequence \((a_n)\) converges to \( y \), or \( \lim_{n \to \infty} a_n = y \) if

\[
\forall k > 0 \exists n \forall m \geq n (|a_m - y| < 1/k),
\]

or equivalently,

\[
\exists N \forall k > 0 \forall m \geq N (k) (|a_m - y| < 1/k).
\]

\( N \) is a witness for convergence, also called a modulus of convergence. If \( N \) is a modulus of convergence for the sequence \((a_n)\), then it is easy to verify that \( (a_{N(2n)}(2n))_n \) is a real number and is the limit of \((a_n)\). Therefore, \( \lim_{n \to \infty} a_n \) can be seen as a term containing \( N \):

\[
\lim_{n \to \infty} a_n \equiv \lambda n. a_{N(2n)}(2n).
\]
Note that, as a term, \( \lim_{n \to \infty} a_n \) contains the modulus of convergence as a subterm, although it is not explicitly shown in the notation.

Similarly, \((a_n)\) is a \textit{Cauchy sequence} if

\[
\forall k > 0 \exists n \forall i, j \geq n \left( |a_i - a_j| < 1/k \right),
\]

or equivalently,

\[
\exists N \forall k > 0 \forall i, j \geq N(k) \left( |a_i - a_j| < 1/k \right),
\]

and \(N\) is a modulus of Cauchyness for \((a_n)\). Then, if \(N\) is a modulus of Cauchyness for \((a_n)\), the sequence \((b_k) \equiv \left( (a_{N(3k)})_{3k} \right)\) of rational numbers is a real number and is the limit of \((a_n)\). Therefore, a sequence has a limit if and only if it is a Cauchy sequence.

Basic theorems regarding limits can be proved. The sum of a infinite series \(\sum_{i=0}^{\infty} a_i\) is defined similarly and basic results regarding series can be proved. These include the common tests for convergence.

A function \(f : [a, b] \to \mathbb{R}\) is \textit{continous} if

\[
\exists \omega \forall x,y \in [a, b] \forall n > 0 \left( |x - y| \leq \omega(n) \rightarrow |f(x) - f(y)| \leq 1/n \right).
\]

\(\omega\) is a witness for continuity, called a \textit{modulus of continuity}. \(C([a, b], \mathbb{R})\) then denotes the set of such continuous functions. As in Bishop’s constructive mathematics, the intermediate value theorem has to take the approximate format: If \(f \in C([a, b], \mathbb{R})\) and \(f(a) < f(b)\), then for any \(y\) such that \(f(a) \leq y \leq f(b)\), and any \(\varepsilon > 0\), there exists \(x \in [a, b]\) such that \(|f(x) - y| < \varepsilon;\)
moreover, if $f$ is strictly increasing, then there exists $x \in [a, b]$ such that $f(x) = y$.

For $I = [a, b]$ a compact interval, $g$ is a derivative of $f$ on $I$, if $g, f \in C(I, \mathbb{R})$, and there exists $\delta$, such that $\delta(n) > 0$ for all $n > 0$, and

$$|f(y) - f(x) - g(x)(y-x)| \leq |y-x|/n$$

for all $x, y \in I$, $|x-y| \leq \delta(n)$. We will use the notations $g = f'$ and $g(x) = \frac{df(x)}{dx}$. $\delta$ is called a modulus of differentiability for $g = f'$. $f$ is differentiable, if there exists $g$ such that $f' = g$. Arbitrary finite order derivatives can then be defined as well.

Now, consider Riemann integration. A finite sequence of real numbers $P = (a_0, ..., a_n)$ is a partition of an interval $I = [a, b]$ if $a = a_0 \leq a_1 \leq ... \leq a_n = b$. Define the Riemann sum

$$S(f, P) = \sum_{i=0}^{n-1} f(a_i)(a_{i+1} - a_i).$$

To define Riemann integration, we choose a sequence of standard partitions $(P_n)$, $P_n \equiv (a, ..., a + \frac{b-a}{2^n}, ..., b)$. It can be proved that if $f \in C(I, \mathbb{R})$ then $(S(f, P_n))_n$ is a Cauchy sequence. Therefore, we let

$$\int_{a}^{b} f(x) \, dx \equiv \lim_{n \to \infty} S(f, P_n).$$

The construction of the limit on the right hand side actually depends on a modulus of Cauchyness for $(S(f, P_n))_n$, which in turn depends on a given modulus of continuity $\omega$ of $f$ on $[a, b]$. So, we should write the limit as a term
including \( f, \omega, a, b \) as parameters. It can be shown that if \( \omega' \) is also a modulus of continuity of \( f \) on \([a, b]\) then 
\[
T[f, \omega, a, b] = \mathbb{R} T[f, \omega', a, b].
\]
So, we can simply use the notation 
\[
\int_{a}^{b} f(x) \, dx.
\]

Basic theorems of calculus then follow, including an approximate form of Rolle’s theorem, Taylor series theorem, and the fundamental theorem of calculus and so on.

5.2. Metric spaces. The development of the theory of metric spaces in strict finitism is a typical case of using the technique of abstraction in strict finitism. The theory of metric spaces is presented as schematic claims about an arbitrary set satisfying some conditions. Recall that a set is a pair of formulas. Therefore, the theory actually consists of schematic claims with any formulas of some formats. Applying this general theory of metric spaces to a more concrete metric space, for instance, the metric space of real numbers or the metric space of continuous functions, means instantiating the definitions and constructions with concrete formulas in those formats. We mostly follow the ideas in Chapter 4 of Bishop and Bridges [5], with necessary modifications to fit into our framework. This shows that abstract mathematics can also be developed within strict finitism.

Suppose that \( X \) is a set. \( \rho \) is a metric on \( X \), or \((X, \rho)\) is a metric space, if \( \rho : X \times X \to \mathbb{R}^+ \) and for all \( x, y, z \in X \), (i) \( \rho(x, y) = 0 \iff x =_X y \), and (ii) \( \rho(x, y) = \rho(y, x) \), and (iii) \( \rho(x, y) \leq \rho(x, z) + \rho(z, y) \). Note that, in strict finitism, ‘\((X, \rho)\) is a metric space’ is a schematic claim containing
two arbitrary formulas defining a set \( X \) and containing an arbitrary term \( \rho \). Therefore, we do not quantify over metric spaces in strict finitism.

Familiar notions can be carried over from those in classical mathematics straightforwardly, for instance: \((X, \rho)\) is bounded, if and only if there exists \( M \) such that \( \rho(x, y) \leq M \) for all \( x, y \in X \); A sequence \((x_n)\) of elements of \( X \) converges to an element \( y \) of \( X \), if and only if for any \( k > 0 \), there exists \( N \), such that \( \rho(x_n, y) \leq 1/k \) for \( n > N \); \( f \) is a uniformly continuous function from a metric space \((X, \rho)\) to another metric space \((X', \rho')\), if and only if for any \( k > 0 \), there exists \( N > 0 \), such that \( \rho'(f(x), f(y)) \leq 1/k \), whenever \( x, y \in X \) and \( \rho(x, y) \leq 1/N \). Note that some of these notions require witnesses. For instance, the bound \( M \) is a witness for the property ‘\((X, \rho)\) is bounded’, and convergence, uniform continuity and so on all need a modulus of convergence or continuity.

Completeness, total boundedness, and compactness of metric spaces can be defined as in Bishop’s constructive mathematics. Completion of a metric space can then be constructed and other theorems in Bishop’s constructive mathematics can be carried over, including the Stone-Weierstrass theorem.

5.3. **Complex analysis.** Basics of complex are developed in a similar manner as calculus. We have Cauchy’s integration theorem, Cauchy’s integral formula, estimates of zeros and maximum values for differentiable functions, and the fundamental theorem of algebra. We will skip the details here.
5.4. **Lebesgue integration.** The definition of Lebesgue integration takes some ideas from Bishop and Bridges [5] with significant changes to simplify the topic and to fit into our more restrictive framework.

Lebesgue integrable functions will be partial functions on \( \mathbb{R} \). Since we cannot quantify over sets, we cannot quantify over arbitrary subsets of \( \mathbb{R} \) as the domains of partial functions. However, we need to quantify over integrable functions on \( \mathbb{R} \). The resolve the problem, we consider sets with parameters. That is, we consider a pair of formulas defining a set but allow the formulas to contain other free variables as parameters. This is then a family of sets and we can quantify over sets in the family by quantifying over parameters. Therefore, to construct Lebesgue integrations, we first define an index set \( \Gamma \) for the family of domains of Lebesgue integrable functions. Let \( C (\mathbb{R}) \subseteq C (\mathbb{R}, \mathbb{R}) \) be the set of functions in \( C (\mathbb{R}, \mathbb{R}) \) with compact supports (i.e. vanishing outside a finite interval). \( \Gamma \) is a set of sequences of functions in \( C (\mathbb{R}) \):

\[
(f_n) \in \Gamma \iff \forall n \ (f_n \in C (\mathbb{R})) \land \left( \sum_{n=0}^{\infty} \int |f_n| \, d\mu \text{ converges} \right).
\]

Then, we define the family \( \mathcal{D} \equiv \{ D_{(f_n)} : (f_n) \in \Gamma \} \) of subsets of \( \mathbb{R} \) indexed by \( \Gamma \):

\[
x \in D_{(f_n)} \iff x \in \mathbb{R} \land \sum_{n=0}^{\infty} |f_n (x)| \text{ converges}.
\]

\( \mathcal{D} \) will be the family of domains of Lebesgue integrable functions.
To define Lebesgue integrable functions, first let $\mathcal{F}(\mathcal{D}, \mathbb{R})$ denote the set of all partial functions from the family $\mathcal{D}$ to $\mathbb{R}$. This is a set defined by:

\[
((f_n), h) \in \mathcal{F}(\mathcal{D}, \mathbb{R}) \equiv_{df} ((f_n) \in \Gamma \land h : D_{(f_n)} \to \mathbb{R}),
\]

\[
((f_n), h_1) =_{\mathcal{F}(\mathcal{D}, \mathbb{R})} ((g_n), h_2) \equiv_{df} (D_{(f_n)} = D_{(g_n)} \land \forall x \in D_{(f_n)} (h_1(x) = h_2(x))).
\]

Then, we can quantify over all partial functions in $\mathcal{F}(\mathcal{D}, \mathbb{R})$ in our formulas, by which we mean a quantification like

\[
\forall i \forall f ((i, f) \in \mathcal{F}(\mathcal{D}, \mathbb{R}) \to .......).
\]

We will say that $D_i$ is the domain of $(i, f)$. Therefore, the domain (actually the parameter of the domain) of a partial function is uniquely determined by the partial function.

Then, we can define the set $L_1 = L_1(\mathbb{R})$ of Lebesgue integrable functions on $\mathbb{R}$:

\[
(((f_n), f) \in L_1)
\]

\[
\equiv ((f_n), f) \in \mathcal{F}(\mathcal{D}, \mathbb{R}) \land
\]

\[
\exists (g_n) \in \Gamma \left( D_{(g_n)} \subseteq D_{(f_n)} \land \forall x \in D_{(g_n)} \left( f(x) = \sum_{n=0}^{\infty} g_n(x) \right) \right).
\]

Therefore, $L_1 \subseteq \mathcal{F}(\mathcal{D}, \mathbb{R})$. For $((f_n), f)$ above, we will simply call $f$ an integrable function and denote it as $f \in L_1$, and we will call $D_{(f_n)}$ the domain of $f$ and denote it as $dmn(f)$. Among the witnesses for $f \in L_1$ is a sequence $(g_n) \in \Gamma$ satisfying the condition in the definition. This will be called a representation sequence of $f$. 


Lebesgue integrable functions are thus essentially sequences of continuous functions (with compact supports) that converge in some way. They are natural extensions of continuous functions. For instance, the characteristic function of the interval \([0, 1]\) cannot be constructed as a function defined on the whole set \(\mathbb{R}\) in strict finitism, because given any real number we cannot decide if it belongs to the interval. The natural definition by cases

\[
f(x) = \begin{cases} 
1, & \text{for } x \in [0, 1], \\
0, & \text{for } x \in (-\infty, 0) \cup (1, \infty)
\end{cases}
\]

results in a partial function defined on \((-\infty, 0) \cup [0, 1] \cup (1, \infty)\) only, which is a subset of \(\mathbb{R}\). However, it is easy to construct a sequence of continuous functions on \(\mathbb{R}\) that approaches to this partial function on \((-\infty, 0) \cup [0, 1] \cup (1, \infty)\), which will imply that this subset is a domain for Lebesgue integrable functions and the partial function \(f\) defined above is Lebesgue integrable. Therefore, Lebesgue integrable functions naturally extend continuous functions so that more functions needed for applications are available to strict finitism.

We can then prove the completeness of Lebesgue integration. We can define \(L_1\) as a metric space and prove the density of \(C(\mathbb{R})\) in \(L_1\) and the completeness of \(L_1\).

A measurable function is defined as a function in \(\mathcal{F}(\mathcal{D}, \mathbb{R})\) that can be approximated by functions in \(C(\mathbb{R})\) in some way. Then, we can define various notions of convergence, such as convergence in measure, almost everywhere convergence, and almost uniform convergence, and we also have the
dominated convergence theorem and other convergence theorems. We can define the space $L_2$ of square integrable functions and prove the density of $C(\mathbb{R})$ in $L_2$ and the completeness of $L_2$.

5.5. Hilbert space. Like the definition of metric spaces, the definitions for linear space, Banach space and Hilbert space are also schematic definitions involving an arbitrary set. Then, we can define familiar notions such as bases, subspaces, linear operators and so on. We can prove that $L_2$ is a Hilbert space with a non-zero orthonormal basis.

Note that we cannot quantify over all subspaces of a Hilbert space. A linear operator defined on a linear subspace of a Hilbert space consists of a specification of a subspace and an operator on the subspace. Claims concerning such linear operators are then schematic claims. We can define self-adjoint operators and prove the spectral theorem for unbounded self-adjoint operators. We can also prove Stone’s theorem.

6. Some Remarks

This work supports the following conjecture to some degree:

**Conjecture of Finitism.** Strict finitism is in principle sufficient for formulating theories and representing proofs and calculations in the ordinary sciences.

Obviously, the amount of applied mathematics developed in the monograph is still very limited. What has been shown is that an impressive part of applied mathematics that appears to be beyond finitism can actually be
developed within strict finitism. Moreover, the technique for developing applied mathematics there also shows that the scope of strict finitism can advance further. On the other side, there are other intuitive reasons supporting the conjecture. First, in almost all branches of the sciences (except for some areas in the fundamental physics, perhaps) we deal with finite and discrete things in the universe from the cosmological scale to the Planck scale. The ratio between these two linear scales is less than $10^{100}$. Linear magnitudes and precisions beyond $10^{100}$ or $10^{-100}$ (for ordinary physics units) are thus physically meaningless. These magnitudes are bounded by the power function. Even if we consider the number of states of a system, the magnitudes are still bounded by an iteration of the power function. Functions not essentially bounded by a few iterations of the power function never appear in common scientific contexts (even in the fundamental physics). These facts intuitively suggest that real numbers, functions of real numbers, and other mathematical entities or structures encoded as elementary recursive functions are perhaps sufficient for the realistic applications in the sciences.

Second, infinity and continuity and so on are approximations to finite and discrete things in the applications. They help us to suppress microscopic details and reduce complexity. Intuitively, they ought not to be strictly indispensable. For instance, we use a differentiable function to represent population growths. If the differentiability condition for a population growth function is strictly logically indispensable for proving a conclusion about the real population growths on the Earth, we have reasons to suspect that the
conclusion is not reliable, because conditions such as differentiability are only approximations at the macro-scale and should not be taken too literally. For instance, some discretized versions of the relevant equations, theorems or proofs ought to be available if the application is to draw a practically meaningful conclusion. Then, we have reasons to expect that the proof is essentially finitistic. Similarly, the Jordan Curve Theorem in its original format may not be available to strict finitism. However, considering the fact that the spacetime structure below the Planck scale is still unknown (and may be discrete or not 4-dimensional), we can expect that if the theorem is applicable in some real situation, what is really relevant for the application must be some version of the theorem that does not take infinity or the continuity of space too literally (e.g. a version where lines have a non-zero minimum width and space regions have a non-zero minimum size). Such a version is likely to be essentially finitistic.

Third, there are indeed beliefs about concrete, finite things that are not obtainable without entertaining advanced abstract mathematical concepts, or without entertaining the axioms implying infinity. For instance, if we design a computer program simulating the proofs in the formal system ZFC, we believe that the program will never output \( 0 = 1 \) as a theorem, which follows from the belief that ZFC is consistent. This belief about a concrete, finite thing is perhaps not obtainable without entertaining the concepts and axioms in set theory. Now, the belief in the consistency of ZFC seems to be inductive in nature. In other words, after we practice entertaining
concepts and constructing proofs in set theory for a long time, and after obvious paradoxes are eliminated, we come to believe that no paradoxes will be derived in the future. This appears to be essentially an inductive belief about what will happen in a type of mental activities, based on our reflections upon (i.e. observing) our own mental activities, and then idealized by ignoring the fact that we may make mistakes and we cannot really perform arbitrarily long inferences. It should not be surprising that such a belief is not obtainable without entertaining those abstract mathematical concepts, because it is just about what will happen in entertaining and manipulating those abstract mathematical concepts.

If this characterization of the belief in the consistency of ZFC is correct, then this example is not a counter example to the conjecture. First of all, the belief about that concrete computer actually follows (finitistically) from an inductive belief about our own mental activities. This inductive belief is the real premise from which we derive our knowledge about that concrete computer. We did not really use the axioms of ZFC as premises in deriving that piece of knowledge. Secondly, such knowledge about concrete things in the universe does not belong to the ordinary sciences. The conjecture is interested in the minimum mathematical system strictly needed for the ordinary sciences only. What the example shows is at most that some of our inductive beliefs about concrete things in this universe may not be derivable from the ordinary scientific laws from physics to biology. It is a different issue whether or not this fact supports a realistic interpretation of classical
mathematics. I believe that the answer is ‘no’, but I cannot discuss it here (but see Ye [11]).

These, of course, are only intuitive reasons and are far from conclusive. More work has to be done in developing applied mathematics in strict finitism, as well as in analyzing what could be a counter-example to the conjecture, in order to get a definite answer to the conjecture. However, based on what has been developed within strict finitism and based on these intuitive reasons, a positive answer to the conjecture seems plausible.

If the conjecture turns out true, one of the philosophical implications will be that the attempt to argue for realism in philosophy of mathematics based on the indispensability of abstract mathematical entities for scientific applications will be in doubt. Whether or not this casts doubts on realism directly is another issue. I cannot discuss it here (but see Ye [13]). Another implication is that while Hilbert’s proof theory program does not succeed, Hilbert’s instrumentalist interpretation of classical mathematics may still be correct. The proof theory program fails because it sets its goal too high. It looks for a once and for all proof that all classical mathematics is conservative over finitism. What we suggest here is instead that the part of mathematics that is actually applied in the sciences is still conservative over finitism. This is a piecemeal approach to demonstrating conservativeness limited to applicable mathematics. It does not look as neat as Hilbert’s proof theory program, but it still supports the idea that classical mathematics is only an instrument for scientific applications.
My own use of strict finitism is for another purpose. This work belongs to a research project pursuing a radically naturalistic philosophy of mathematics. See Ye [11] for an introduction to the project. According to this philosophy, human mathematical practices are human brains’ cognitive activities, and what really exist in human mathematical practices are human brains and mathematical concepts and thoughts inside brains realized as neural circuitries (and there are no alleged abstract mathematical entities ‘outside the brains’). Then, accounting for classical mathematics consists in describing the cognitive functions of classical mathematical concepts and thoughts inside brains. In particular, the applicability of classical mathematics means some natural regularity among the natural phenomena of mathematical practices and applications of human brains, that is, the fact that some brain processes of applying mathematics always produce some results in some normal situations. Then, explaining applicability becomes explaining some regularity among a class of natural phenomena, which is a completely scientific task. By some abstractions to ignore physical, physiological and psychological details, explaining applicability becomes a logical problem.

The main obstacle for a logical explanation of applicability is then recognized to be the fact that classical mathematical concepts and thoughts appear to be ‘about infinite entities’, and therefore they cannot be translated, without changing logical structures, into literally true assertions about the subject matter of applications, which is always finite and discrete for the
strict finitism is designed as an asistant tool for explaining applicability. The idea is that since applied mathematics can be developed within strict finitism, the applications of classical mathematics to finite and discrete things in the universe can in principle be reduced to the applications of strict finitism. The applications of strict finitism can be interpreted as valid logical deductions from literally true premises about strictly finite concrete things in the universe, to literally true conclusions about those finite things. For instance, applying strict finitism to a physics system $X$ can be interpreted as using a computer $C$ to encode information about $X$ and simulate $X$. Mathematical theorems in strict finitism are interpreted as assertions about $C$. Physics laws describe $X$ by stating how $C$ encodes information about $X$ and simulates $X$. Initial observation data are assertions directly about $X$. These are literally true premises about strictly finite concrete things in the universe. An application of strict finitism actually logically derives a conclusion about $X$ from these premises. This then will provide a logically transparent explanation of why infinite mathematics can be applied to derive literal truths about strictly finite things in the universe. This is a very brief summary of the idea for explaining applicability within the project. Another paper (Ye [14]) explains the philosophical aspect of this approach to explaining applicability. Both philosophical and technical details are given in the monograph (Ye [12]).
Finally, this work may look similar to several nominalization programs in philosophy of mathematics. (See Burgess and Rosen [6] for a survey.) On the technical side, this work differs from them mainly in that the basis here, strict finitism, is the most restrictive one. There are two motivations for this restriction. First, only a strictly finitistic system can be an assistant tool for explaining the applicability of classical mathematics to strictly finite things in the universe. Second, committing to reality of infinity in any format (including potential infinity) in philosophy of mathematics will mean committing to things that may not belong to this physical universe. It will thus violate the principle of nominalism, and it will face the well-known Benacerrafian epistemological difficulty as realism does (Benacerraf [3]). In other words, nominalization programs that commit to infinity (even if a potential infinity only) may not be coherent nominalism. I will not discuss the details here (but see Ye [13]). Besides, my own philosophical position is a radical naturalism or physicalism, which is different from the positions held by those who pursue nominalization programs. (See Ye [11].)

References


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