Epistemic Logic IV:
from friends to couples

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Andreas Herzig’s classification of logic and action

<table>
<thead>
<tr>
<th>no knowledge</th>
<th>knowledge</th>
<th>group</th>
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<tbody>
<tr>
<td>no action</td>
<td>PL</td>
<td>EL</td>
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<tr>
<td>action</td>
<td>PDL, TL</td>
<td>ETL, DEL, EPDL</td>
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<tr>
<td>strategy</td>
<td>ATL, STIT</td>
<td>AETL, ESTIT</td>
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</tbody>
</table>

The different levels of rationality (rows):

- reason logically
- act cleverly
- interact intelligently
- everything above under uncertainty
No knowledge but action or time

No knowledge but agent-based strategy

Combinations
Propositional dynamic logic

Propositional dynamic logic (where \( a \in A \)):

\[
\begin{align*}
\phi &::= T \mid p \mid \neg \phi \mid (\phi \land \phi) \mid [\pi] \phi \\
\pi &::= a \mid ?\phi \mid (\pi; \pi') \mid (\pi + \pi) \mid \pi^*
\end{align*}
\]

\([\pi] \phi\) reads: \( \phi \) holds after any successful execution of program \( \pi \).

\([\text{while } \phi \text{ do } a] \psi := [(?\phi; a)^*; ?\neg \phi] \psi, \ C_{\{a,b\}} \phi := [(a + b)^*] \phi\) etc.

A model is a tuple \( \langle S, \{\rightarrow | a \in A\}, V \rangle \) the semantics is given by:

\[
\begin{array}{rcl}
M, s \models [\pi] \phi &\iff& \forall t : sR_\pi t \text{ implies } M, t \models \phi \\
R_a &=& \rightarrow \\
R_{?\phi} &=& \{(s, s) \mid M, s \models \phi\} \\
R_{\pi;\pi'} &=& R_\pi \circ R_{\pi'} \\
R_{\pi+\pi'} &=& R_\pi \cup R_{\pi'} \\
R_{\pi^*} &=& \bigcup_{n \in \mathbb{N}} R_\pi^n
\end{array}
\]
Important axioms

- Distribution axioms and necessitation rules for $\lbrack \pi \rbrack$
- $[\lnot \phi]p \iff (\phi \rightarrow p)$
- $[\pi ; \pi']p \iff [\pi][\pi']p$
- $[\pi + \pi']p \iff [\pi]p \land [\pi']p$
- Fixed Point: $[\pi^*]p \iff (p \land [\pi][\pi^*]p)$
- Induction: $(p \land [\pi^*](p \rightarrow [\pi]p)) \rightarrow [\pi^*]p
Temporal logic

Linear-time temporal logic:

$$\phi ::= \top \mid p \mid \neg \phi \mid (\phi \land \phi) \mid X\phi \mid (\phi U \phi)$$

A model is $\langle R, V \rangle$ where:

- $R$ is a non-empty set of runs (intuitively, infinite sequences indexed by natural numbers, labelled by propositions);
- $V : R \times \mathbb{N} \rightarrow 2^P$.

$$M, (r, t) \models X \phi \iff M, (r, t + 1) \models \phi$$

$$M, (r, t) \models \phi U \psi \iff \exists t' \geq t \in \mathbb{N} \text{ such that } M, (r, t') \models \psi$$

and $\forall t'' : t \leq t'' \leq t' : M, (r, t'') \models \psi$

$$F \phi ::= \top U \phi, \ G \phi ::= \neg F \neg \phi, \ \phi W \psi ::= (\phi U \psi) \lor G \phi$$
Important axioms to axiomatize the logic

- Distribution axioms and necessitation rules for $X$ and $G$
- Determinacy: $X\neg p \iff \neg Xp$
- Fixed Point: $Gp \iff (p \land XGp)$
- Induction $G$: $p \land G(p \rightarrow Xp) \rightarrow Gp$
- Induction $U$: $pUq \iff q \lor (p \land X(pUq))$
- Interaction: $pUq \rightarrow Fq$
Branching-time temporal logic

Computational tree logic (CTL):

\[ \phi ::= \top | p | \neg \phi | (\phi \land \phi) | EX\phi | EG\phi | E(\phi U \phi) \]

\(E\) is a path quantifier. \(EF\phi := E[\top U \phi]\), \(AX\phi := \neg EX(\neg \phi)\), \(AG\phi := \neg EF(\neg \phi)\), etc. Note that \(EG\) is not expressible by \(EU\).

It is interpreted on a transition system \( \langle S, \rightarrow, V \rangle \) where \(\rightarrow\) is serial.

Here is the \textit{rough} idea for the semantics:

\[
\begin{align*}
M, s \vDash EX\phi &\iff \exists r \text{ starting at } s \text{ such that } M, (r, 0) \vDash X \phi \\
M, s \vDash EG\phi &\iff \exists r \text{ starting at } s \text{ such that } M, (r, 0) \vDash G \phi \\
M, s \vDash E(\phi U \psi) &\iff \exists r \text{ starting at } s \text{ such that } M, (r, 0) \vDash \phi U \psi
\end{align*}
\]

More precisely, e.g.,

\[
\begin{align*}
M, s_0 \vDash EG\phi &\iff \exists \text{ a path } s_0 s_1 s_2 \ldots \text{ such that } \forall k \in \mathbb{N} : M, s_k \vDash \phi
\end{align*}
\]

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Some important axioms and rules

Fixed point axioms:
- $E(pUq) \iff q \lor (p \land EXE(pUq))$
- $A(pUq) \iff q \lor (p \land AXA(pUq))$

Induction Rules:
- from $r \rightarrow (\neg q \land EXr)$ infer $r \rightarrow \neg A(pUq)$
- from $r \rightarrow (\neg q \land AX(r \lor \neg E(pUq)))$ infer $r \rightarrow \neg E(pUq)$
To compare LTL and CTL

We can also define LTL semantics over pointed Kripke models:

\[ \mathcal{M}, s \models \phi \iff \forall \text{ path } r \text{ starting from } s : \mathcal{M}, (r, 0) \models \phi \]

- Model checking problems of LTL and CTL on finite models are decidable \((m = |\mathcal{M}|, n = |\phi|)\):
  - CTL: \(O(mn)\) using labelling and fixed-point computation
  - LTL: \(O(m2^n)\) using emptiness checking of Büchi automata
- LTL is more intuitive to express fairness conditions: \(GFp\)
- LTL-formulae can be \textit{exponentially shorter} than their equivalents in CTL.
LTL and CTL are not comparable in expressibility:

- $(A)FGp$ is not expressible in CTL.
- $AG(EFp)$ is not expressible in LTL.

They are fragments of $CTL^*$ where the two quantifiers can be composed arbitrarily (not sticking to each other).
Alternating-time temporal logic [Alur et al. 1997]

$$\phi ::= T \mid p \mid \neg \phi \mid (\phi \land \phi) \mid \langle A \rangle X \phi \mid \langle A \rangle G \phi \mid \langle A \rangle (\phi U \phi)$$

The model is called *concurrent game structure* (given a set of agents $I$ and a set of actions $A$):

$$\mathcal{M} = \langle S, d, \delta, V \rangle$$

where:

- $d : S \times I \to 2^A$ gives available actions for each agent;
- $\delta : S \times A^I \to S$ is a partial transition function coherent with $d$. 
Semantics for ATL

A strategy for an agent is a function: $S^+ \rightarrow A$ coherent with the available actions. A collective strategy for a group $A \subseteq I$ is a function: $S^+ \times A \rightarrow A$ where $S^+$ is the set of non-empty finite sequences of the states in $S$.

The semantics is given by:

\[
\begin{align*}
\mathcal{M}, s \models \langle A \rangle X \phi & \iff \text{there is a collective strategy } \eta \text{ such that: for every path } r \text{ w.r.t. } \eta: \mathcal{M}, r[1] \models \phi \\
\mathcal{M}, s \models \langle A \rangle G \phi & \iff \text{there is a collective strategy } \eta \text{ such that: for every path } r \text{ w.r.t. } \eta: \text{ for all } k: \mathcal{M}, r[k] \models \phi \\
\mathcal{M}, s \models \langle A \rangle \phi U \psi & \iff \text{there is a collective strategy } \eta \text{ such that: for every path } r \text{ w.r.t. } \eta: \exists t' \geq t \in \mathbb{N} \text{ such that: } \mathcal{M}, r[t'] \models \psi \text{ and } \forall t \leq t'' \leq t': \mathcal{M}, r[t''] \models \phi
\end{align*}
\]
Important axioms

- \( \neg \langle A \rangle X \bot \)
- \( \langle A \rangle X \top \)
- \( \neg \langle \emptyset \rangle X \neg p \rightarrow \langle A \rangle X p \)
- \( \langle A_1 \rangle X p \land \langle A_2 \rangle X q \rightarrow \langle A_1 \cup A_2 \rangle X (p \land q) \) (disjoint \( A_1 \) and \( A_2 \))
- \( \langle A \rangle G p \leftrightarrow (p \land \langle A \rangle X \langle A \rangle G p) \)
- \( \langle \emptyset \rangle G (q \rightarrow (p \land \langle A \rangle X q)) \rightarrow \langle \emptyset \rangle G (q \rightarrow \langle A \rangle G p) \)
- \( \langle A \rangle p U q \leftrightarrow q \lor (p \land \langle A \rangle X \langle A \rangle p U q) \)
- \( \langle \emptyset \rangle G ((q \lor (p \land \langle A \rangle X r)) \rightarrow r) \rightarrow \langle \emptyset \rangle G (\langle A \rangle p U q \rightarrow r) \)
• ATL extends CTL: $A := \langle \emptyset \rangle$, $E := \langle I \rangle$.

• The model-checking problem for ATL is PTIME-complete, and can be solved in time $O(|\mathcal{M}| \cdot |\phi|)$ by fixed-point computation.

• Strategy for ATL can be synthesized incrementally.

• Model checking ATL formulas $\langle A \rangle \phi$ corresponds to solving concurrent extensive games.

Problem: $\langle i \rangle G(married \land \langle i \rangle X \neg married)$ is satisfiable! (strategy may be changed)
See to it that

The language of STIT logic:

\[ \phi ::= T | p | \neg \phi | (\phi \land \phi) | \Box \phi | [i:\text{stit}] \phi \]

A model is \( \langle T, <, C, V \rangle \) where:

- \( \langle T, < \rangle \) is a tree
- \( C \) gives for each \( i \) each \( m \in T \) a partition over histories through \( m \), representing the choices. Call the induced equivalence relation \( R^m_i \). We require that the choices of agent at a moment always intersect.
- \( V \) assigns to each \( (h, m) \) a set of basic propositions

\[
\begin{align*}
\mathcal{M}, (h, m) \models \Box \phi & \iff \forall h' : m \in h' \text{ implies } \mathcal{M}, (h', m) \models \phi \\
\mathcal{M}, s \models [i:\text{stit}] \phi & \iff \forall h' : (h, m)R^m_i(h', m) \text{ implies } \mathcal{M}, (h', m) \models \phi
\end{align*}
\]

It can be extended to \([A:\text{stit}] \phi\) where we consider the intersection of \( R^m_i \). It can also be extended with temporal operators.
Important axioms

- S5 for $\Box$
- S5 for $[i \text{ stit}]$
- $\Box \phi \rightarrow [i \text{ stit}]\phi$
- $\Diamond [i_1 \text{ stit}]\phi_1 \land \cdots \land [i_n \text{ stit}]\phi_n \rightarrow \Diamond ([i_1 \text{ stit}]\phi_1 \land \cdots \land [i_n \text{ stit}]\phi_n)$
  the agents’ choices are independent.

Deliberative STIT: $[i \text{ stit}]\phi \land \neg \Box \phi$.

At each point $(h, m)$, each player’s choice (the set of pairs which includes $(h, m)$) is fixed. STIT logic can handle the (true) agency: certain consequence is due to certain (choice of the) agent.
Combinations

Knowledge comes in if there is uncertainty:

- Epistemic temporal logic (ETL: linear /branching)
- Dynamic epistemic logic (with PDL program) (next lecture)
- Alternating-time temporal epistemic logic (ATEL)
- Epistemic see-to-it-that logic (ESTIT)
Epistemic temporal logic

Linear-time temporal logic:

$$\phi ::= T \mid p \mid \neg \phi \mid (\phi \land \phi) \mid X \phi \mid (\phi U \phi) \mid K_i \phi$$

A model is $\langle R, \sim, V \rangle$ where:

- $R$ is a non-empty set of runs;
- $\sim: I \rightarrow 2^{Points \times Points}$ where $Points = R \times \mathbb{N}$ such that $\sim_i$ is an equivalence relation;
- $V: Points \rightarrow 2^P$.

$$\mathcal{M}, (r, t) \models K_i \phi \iff \forall (r', t') \sim_i (r, t): \mathcal{M}, (r', t') \models \phi$$
Different properties about knowledge and time

- Unique initial state: \( r[0] = r'[0] \) for all \( r, r' \in R \);
- Synchrony: for all points \( (r, m) \) and \( (r', n) \) if \( (r, m) \sim_i (r', n) \) then \( m = n \);
- Perfect recall: for all points \( (r, m) \sim_i (r', n) \), if \( m > 0 \) then either \( (r, m-1) \sim_i (r', n) \) or there exists \( l < n \) such that \( (r, m-1) \sim_i (r', l) \) and for all \( l < k \leq n \): \( (r, m) \sim_i (r', k) \).
- No learning: for all points \( (r, m) \sim_i (r', n) \), either \( (r, m+1) \sim_i (r', n) \) or there exists \( l > n \) such that \( (r, m+1) \sim_i (r', l) \) and for all \( n \leq k < l \): \( (r, m) \sim_i (r', k) \).

Idea: PR agents can only refine the information cells, NL agents can only make them more coarse (not learning).
Example of perfect recall
Complexity: depending on the assumptions

The logics are computationally quite different (multi-agent no CK):

- $\text{PSPACE}$-complete (none, sync, sync+uis, uis)
- $\text{EXPSPACE}$-complete (nl+sync+uis, nl+pr+sync+uis)
- non-elementary time (pr, pr+sync, pr+uis, pr+sync+uis)
- non-elementary space (nl, nl+pr, nl+pr+sync, nl+sync)
- not decidable (nl+uis, nl+pr+uis)

Model checking is usually given by finitely generated interpreted systems using local states.
Important axioms

To axiomatize logics:

- **S5+LTL** \((\text{none, sync, sync+uis, uis})\)
- **KT2**: \(K_i X p \rightarrow XK_i p\) \((\text{pr+sync, pr+sync+uis})\)
- **KT3**: \(K_i p \wedge X(K_i q \wedge \neg K_i r) \rightarrow \neg K_i \neg(K_i p U(K_i q U \neg r))\) \((\text{pr, pr+uis})\)
- **KT4**: \((K_i p U K_i q) \rightarrow K_i(K_i p U K_i q)\) \((\text{nl})\)
- **KT5**: \(XK_i p \rightarrow K_i X p\) \((\text{nl+sync})\)
- **KT6**: \(K_i p \leftrightarrow K_1 p\)

Combination to axiomatize logics:

- **KT2+5** \((\text{nl+pr+sync})\)
- **KT2+5+6** \((\text{nl+sync+uis, nl+pr+uis})\)
- **KT3+4** \((\text{nl+pr, nl+pr+uis, nl+uis})\)
Alternating-time epistemic logic

What if we just add some epistemic relation in the model as in the temporal logic case?

Something wrong $K_i\langle 1 \rangle X p$ holds at any state of the first level (de dicto vs. de re again).
The strategy has to be uniform to be executable!
Possible solution in epistemic STIT

- By inserting the knowledge operator at the right position.
- $\langle i \rangle X \phi$ in temporal STIT logic is $\Diamond [i \text{ stit}] X \phi$ (there is a choice that $i$ can make sure...)
- $K_i \langle 1 \rangle X p = K_i \Diamond [i \text{ stit}] X \phi$
- $\Diamond K_i [i \text{ stit}] X \phi$ looks more promising.