A Super Introduction to Reverse Mathematics

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Outline

- Background
- Second Order Arithmetic
- \( \text{RCA}_0 \) and Mathematics in \( \text{RCA}_0 \)
- Other Important Subsystems
- Reverse Mathematics and Other Branches
- References
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- References
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Reverse mathematics is a program in mathematical logic that seeks to determine which axioms are required to prove theorems of mathematics.

The program was founded by Harvey Friedman (1975, 1976). A standard reference for the subject is Simpson’s (2009).

The object of reverse mathematics is non-set theoretic or ordinary. The distinction between set-theoretic and ordinary mathematics corresponds roughly to the distinction between “uncountable mathematics” and “countable mathematics”.

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The object of reverse mathematics is non-set theoretic or ordinary. The distinction between set-theoretic and ordinary mathematics corresponds roughly to the distinction between “uncountable mathematics” and “countable mathematics”.
To show that a system $S$ is required to prove a theorem $T$, two proofs are required. The first proof shows $T$ is provable from $S$; this is an ordinary mathematical proof along with a justification that it can be carried out in the system $S$. The second proof, known as a reversal, shows that $T$ itself implies $S$; this proof is carried out in the base system. The reversal establishes that no axiom system $S'$ that extends the base system can be weaker than $S$ while still proving $T$. 
The language of second order arithmetic($L_2$) is a two-sorted language. This means that there are two distinct sorts of variables which are intended to range over all natural numbers and all subsets of natural numbers. The first sort are called number variables, denoted $i, j, k, m, n$, the other are called set variables, denoted $X, Y, Z$. What’s more, the language contains 2-nary functions $+$ and $\cdot$, constants 0 and 1 and a order $\lt$.

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Atomic formulas are $t_1 = t_2$, $t_1 < t_2$ and $t_1 \in X$ where $t_1$, $t_2$ are numerical terms and $X$ is any set variable. Formulas are built up from atomic formulas by means of propositional connectives and number quantifiers $\forall n$, $\exists n$, and set quantifiers $\forall X$, $\exists X$. 
Second Order Arithmetic ($\mathbb{Z}_2$)

- $L_2$-structures. A model for $L_2$ is an ordered 7-tuple

$$M = (|M|, S(M), +_M, \cdot_M, 0_M, 1_M, <_M)$$

Where $|M|$ is a set which serves as the range of the numbers, $S(M)$ is a set of subsets of $|M|$ serving as the range of the set variables. $+_M$ and $\cdot_M$ are binary operations on $|M|$, $0_M$ and $1_M$ are distinguished elements of $|M|$, and $<_M$ is binary relation on $|M|$. We always assume that the sets $|M|$ and $S(M)$ are disjoint and nonempty.

- Parameters. Let $B$ be any subset of $|M| \cup S(M)$. By a formula with parameters from $B$ we mean a formula of the extended language $L_2(B)$. Here $L_2(B)$ consists of $L_2$ augmented by new constant symbols corresponding to the elements of $B$. 
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- Parameters. Let $\mathcal{B}$ be any subset of $|M| \cup S(M)$. By a formula with parameters from $\mathcal{B}$ we mean a formula of the extended language $L_2(\mathcal{B})$. Here $L_2(\mathcal{B})$ consists of $L_2$ augmented by new constant symbols corresponding to the elements of $\mathcal{B}$. 
Definable. A set $A \subseteq |M|$ is said to be definable over $M$ allowing parameters from $B$ if there exists a formula $\varphi(a)$ with parameters from $B$ and no free variables other than $n$ such that

$$A = \{ a \in |M| : M \models \varphi(a) \}$$

Here $M \models \varphi(a)$ means that $M$ satisfies $\varphi(a)$, i.e. $\varphi(a)$ is true in $M$. 

(i) Basic Axioms:

\[ \forall m (m + 1 \neq 0), \]
\[ \forall m (m \cdot 0 = 0), \]
\[ \forall m (m \cdot (n + 1) = m \cdot n + m), \]
\[ \forall m (m + n + 1 = (m + n) + 1), \]
\[ \forall m (m < n + 1 \leftrightarrow (m = n \lor m < n)). \]

(ii) Induction Axiom:

\[ (0 \in X \land \forall n (n \in X \rightarrow n + 1 \in X)) \rightarrow \forall n (n \in X). \]
(iii) Comprehension Scheme

\[ \exists X \forall n (n \in X \iff \varphi(n)) \]

where \( \varphi(n) \) is any formula of \( L_2 \) in which \( X \) does not occur freely.

Intuitively, the given instance of comprehension scheme says that there exists a set \( X = \{ n : \varphi(n) \} \) = the set of all \( n \) such that \( \varphi(n) \) holds. This set is said to be definable by the given formula \( \varphi(n) \).

In the comprehension scheme, \( \varphi(n) \) may contain free variable in addition to \( n \). These free variables may be referred to as parameters of this instance of the comprehension scheme.
Second Order Arithmetic ($Z_2$)

- $Z_2$ is strong enough to develop analysis.
- If $T$ is any subsystem of $Z_2$, a model of $T$ is any $L_2$-structure satisfying the axioms of $T$. By Gödel’s completeness theorem applied to the two sorted language $L_2$. We have the following important principle: a given $L_2$-sentence $\sigma$ is a theorem of $T$ if and only if all model of $T$ satisfies $\sigma$.
- We shall see that subsystems of $Z_2$ provide a setting in which the Main Question can be investigated in a precise and fruitful way.
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Z₂ is strong enough to develop analysis.

If \( T \) is any subsystem of \( Z₂ \), a model of \( T \) is any \( L₂ \)-structure satisfying the axioms of \( T \). By Gödel’s completeness theorem applied to the two sorted language \( L₂ \). We have the following important principle: a given \( L₂ \)-sentence \( \sigma \) is a theorem of \( T \) if and only if all model of \( T \) satisfies \( \sigma \).

We shall see that subsystems of \( Z₂ \) provide a setting in which the Main Question can be investigated in a precise and fruitful way.
Recursive Comprehension Axiom (RCA). The RCA scheme consists of all formulas of the form

$$\forall n (\varphi(n) \leftrightarrow \psi(n)) \rightarrow \exists X \forall n (n \in X \leftrightarrow \varphi(n))$$

where $\varphi(n)$ is any $\Sigma^0_1$ formula, $\psi(n)$ is any $\Pi^0_1$, $n$ is any number variable, and $X$ is a set variable which does not occur freely in $\varphi(n)$.

In the RCA, note that $\varphi(n)$ and $\psi(n)$ may contain parameters, i.e., free set variables and free number variables in addition to $n$. Thus all $L_2$-structure satisfies RCA if and only if $S(M)$ contains all subsets of $|M|$ which are $\Delta^0_1$ definable over $M$ allowing parameters from $|M| \cup S(M)$. 
RCA₀ is the subsystems of Z₂ consisting of the basic axioms, the $\Sigma^0_1$ induction scheme, and the RCA scheme.

The system RCA₀ plays two key roles in Reverse Mathematics. First, the development of ordinary mathematics within RCA₀ correspond roughly to the positive content of what is known as “computable mathematics” or “recursive analysis”. Thus RCA₀ is a kind of formalized recursive mathematics. Second, RCA₀ frequently play the role of a weak base theory in Reverse Mathematics. Most of the results of Reverse Mathematics will be stated formally as theorems of RCA₀.
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Within RCA\(_0\), we define a paring map \((i, j) = (i + j)^2 + i\), where of course \(i^2 = i \cdot i\).

Within RCA\(_0\), a finite sequence of natural numbers is a finite set \(X\) such that \(\forall n (n \in X \rightarrow \exists i \exists j (n = (i, j)))\) and \(\forall i \forall j \forall k ((i, j) \in X \land (i, k) \in X \rightarrow j = k)\) and \(\exists l \forall i (i < l \leftrightarrow \exists j ((i, j) \in X))\).

Function. The following definitions are made in RCA\(_0\). Let \(X\) and \(Y\) be sets of natural numbers. We write \(X \subseteq Y\) to mean \(\forall n (n \in X \rightarrow n \in Y)\).

We define \(X \times Y\) to be the set of all \(k\) such that \(\exists i \leq k \exists j \leq k (i \in X \land j \in Y \land (i, j) = k)\).

We define a function \(f : X \rightarrow Y\) to be a set \(f \subseteq X \times Y\) such that \(\forall i \forall j \forall k (((i, j) \in f \land (i, k) \in f) \rightarrow j = k)\) and \(\forall i \exists j (i \in X \rightarrow (i, j) \in f)\).
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Lemma 1 (Composition)

The following is provable in RCA\(_0\) If \(f : X \rightarrow Y\) and \(g : Y \rightarrow Z\) then there exists \(h = g \circ f : X \rightarrow Z\) defined by \(h(i) = g(f(i))\).
Lemma 1 (Composition)

The following is provable in RCA$_0$ If $f : X \to Y$ and $g : Y \to Z$ the there exists $h = g \circ f : X \to Z$ defined by $h(i) = g(f(i))$.

Within RCA$_0$, the set of all $s \in \text{Seq}$ such that $lh(s) = k$ is denoted $\mathbb{N}^k$. This set exists by $\Sigma^0_0$ comprehension. If $f : \mathbb{N}^k \to \mathbb{N}$ and $s = \langle n_1, ..., n_k \rangle \in \mathbb{N}^k$, we sometimes write $f(n_1, ..., n_k)$ instead of $f(s)$. 
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Lemma 2 (Primitive recursion)

The following is provable in RCA\(_0\). Given \( f : \mathbb{N}^k \rightarrow \mathbb{N} \) and \( g : \mathbb{N}^{k+2} \rightarrow \mathbb{N} \), there exists a unique \( h : \mathbb{N}^{k+1} \rightarrow \mathbb{N} \) defined by

\[
\begin{align*}
  h(0, n_1, \ldots, n_k) &= f(n_1, \ldots, n_k) \\
  h(m, n_1, \ldots, n_k) &= g(h(m, n_1, \ldots, n_k), m, n_1, \ldots, n_k).
\end{align*}
\]
Lemma 3 (Minimization)

The following is provable in RCA$_0$. Let $f : \mathbb{N}^{k+1} \to \mathbb{N}$ be such that for all $\langle n_1, \ldots, n_k \rangle \in \mathbb{N}^k$ there exists $m \in \mathbb{N}$ such that $f(m, n_1, \ldots, n_k) = 1$. Then there exists $g : \mathbb{N}^k \to \mathbb{N}$ defined by $g(n_1, \ldots, n_k) =$ least $m$ such that $f(m, n_1, \ldots, n_k) = 1$. 

Bounded $\Sigma^0_k$ comprehension. For each $k \in \omega$ the scheme of bounded $\Sigma^0_k$ comprehension consists of all axioms of the form

$$\forall n \exists X \forall i (i \in X \leftrightarrow (i < n \land \phi(i)))$$

where $\phi(i)$ is any $\Sigma^0_k$ formula in which $X$ does not occur freely.

Theorem 4

RCA$_0$ proves bounded $\Sigma^0_1$ comprehension.
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Lemma 3 (Minimization)

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where \( \varphi(i) \) is any \( \Sigma^0_k \) formula in which \( X \) does not occur freely.

Theorem 4

\( \text{RCA}_0 \) proves bounded \( \Sigma^0_1 \) comprehension.
Within RCA₀ we can define \( \mathbb{N}, \mathbb{Z}, \mathbb{Q} \) obviously.

A sequence of rational numbers is defined in RCA₀ to be a function \( f : \mathbb{N} \to \mathbb{Q} \). We usually denote such a sequence as \( \langle q_k : k \in \mathbb{N} \rangle \) where \( q_k = f(k) \).

A real number is defined in RCA₀ to be a sequence of rational numbers \( \langle q_k : k \in \mathbb{N} \rangle \) such that \( \forall k \forall i (|q_k - q_{k+i}| \leq 2^{-k}) \). Two real numbers \( \langle q_k : k \in \mathbb{N} \rangle \) and \( q'_k : k \in \mathbb{N} \) are said to be equal if \( \forall k (|q_k - q'_k| \leq 2^{-k+1}) \).

The sum of two real numbers \( x = \langle q_k : k \in \mathbb{N} \rangle \) and \( y = \langle q'_k : k \in \mathbb{N} \rangle \) is defined by

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x + y = \langle q_{k+1} + q'_{k+1} : k \in \mathbb{N} \rangle
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We note that
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|q_{k+1} + q'_{k+1} - (q_{k+i+1} + q'_{k+i+1})| \\
\leq |q_{k+1} - q_{k+i+1}| + |q'_{k+1} - q'_{k+i+1}| \leq 2^{-k-1} + 2^{-k-1} = 2^{-k},
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so \( x + y \) is a real number.
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Mathematics in RCA\(_0\)

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Mathematics in RCA$_0$

- Trivially $-x = \langle -q_k : k \in \mathbb{N} \rangle$.
- We define $x \leq y$ if and only if $\forall k (q_k \leq q'_k + 2^{-k+1})$.
- It is straightforward to verify in RCA$_0$ that system $(\mathbb{R}, +, -, 0, 1, <)$ obey all the axioms for an ordered Abelian group. Note that formulas such as $x \leq y$, $x = y$, $x + y = z$ are $\Pi^0_1$ while $x < y$, $x \neq 0$, ... are $\Sigma^0_1$.
- Multiplication of real numbers $x = \langle q_k : k \in \mathbb{N} \rangle$ and $y = \langle q'_k : k \in \mathbb{N} \rangle$ is defined by

$$x \cdot y = \langle q_{n+k} \cdot q'_{n+k} : k \in \mathbb{N} \rangle$$

where $n$ is as small as possible such that $2^n \geq |q_0| + |q'_0| + 2$. It is easy to verify that $x \cdot y$ is a real number.
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We define \(x \leq y\) if and only if \(\forall k (q_k \leq q'_k + 2^{-k+1})\).

It is straightforward to verify in RCA\(_0\) that system \((\mathbb{R}, +, -, 0, 1, <)\) obey all the axioms for an ordered Abelian group. Note that formulas such as \(x \leq y, x = y, x + y = z\) are \(\Pi^0_1\) while \(x < y, x \neq 0, \ldots\) are \(\Sigma^0_1\).

Multiplication of real numbers \(x = \langle q_k : k \in \mathbb{N}\rangle\) and \(y = \langle q'_k : k \in \mathbb{N}\rangle\) is defined by

\[
x \cdot y = \langle q_{n+k} \cdot q'_{n+k} : k \in \mathbb{N}\rangle
\]

where \(n\) is as small as possible such that \(2^n \geq |q_0| + |q'_0| + 2\).

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Mathematics in RCA$_0$

- Trivially $-x = \langle -q_k : k \in \mathbb{N} \rangle$.
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- Multiplication of real numbers $x = \langle q_k : k \in \mathbb{N} \rangle$ and $y = \langle q'_k : k \in \mathbb{N} \rangle$ is defined by

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Arithmetical formulas. A formula of $L_2$, or more generally a formula of $L_2(|M| \cup S(M))$ where $M$ is any $L_2$-structure, is said to be arithmetical if it contains no set quantifiers, i.e., all of the quantifiers appearing in the formula are number quantifiers.

Arithmetical comprehension. The arithmetical comprehension scheme is the restriction of the comprehension scheme to arithmetical formulas $\varphi(n)$. Thus we have the universal closure of

$$\exists X \forall n (n \in X \leftrightarrow \varphi(n))$$

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ACA\textsubscript{0} is the subsystem of Z\textsubscript{2} whose axioms are the arithmetical comprehension scheme, the induction axiom and the basic axioms.

The first order arithmetic(Z\textsubscript{1}) is sometimes known as Peano Arithmetic(PA), let $L_1$ be the language of Z\textsubscript{1}. It’s easy to see that for any $L_1$-sentence $\sigma$, $\sigma$ is a theorem of ACA\textsubscript{0} if and only if $\sigma$ is a theorem of Z\textsubscript{1}. In other wards, for any $L_1$-sentence, ACA\textsubscript{0} is a conservative extension of first order arithmetic.

ACA\textsubscript{0} is strong to discuss sequential compactness, countable vector spaces, maximal ideals in countable commutative rings, countable abelian groups and Ramsey’s theorem.
ACA₀ is the subsystem of Z₂ whose axioms are the arithmetical comprehension scheme, the induction axiom and the basic axioms.

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Trees. Within RCA₀ we let

\[
Seq = \omega^{<\omega} = \bigcup_{k \in \omega} \omega^k
\]

denote the set of (codes for) finite sequences of natural numbers. For \( \sigma, \tau \in \omega^{<\omega} \), there is \( \sigma \triangleleft \tau \in \omega^{<\omega} \) which is the concatenation, \( \sigma \) followed by \( \tau \).

A tree is a set \( T \subseteq \omega^{<\omega} \) such that any initial segment of a sequence in \( T \) belongs to \( T \).

A path or infinite path through \( T \) is a function \( f : \omega \to \omega \) such that for all \( k \in \omega \), the initial sequence

\[
f[k] = \langle f(0), f(1), \ldots, f(k-1) \rangle
\]

belong to \( T \).
Weak König’s Lemma. The following definitions are made in RCA\(_0\). We use \(\{0, 1\}^{<\omega}\) or \(2^{<\omega}\) to denote the full binary tree. Weak König’s lemma is the following statement: Every infinite subtree of \(2^{<\omega}\) has an infinite path.

\(\text{WKL}_0\) is defined to be the subsystem of \(\mathbb{Z}_2\) consisting of \(\text{RCA}_0\) plus weak König’s lemma.

In fact, \(\text{WKL}_0\) is strong enough to prove many well known nonconstructive theorems that are extremely important for mathematical practice but not probable in \(\text{RCA}_0\).
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• Weak König’s Lemma. The following definitions are made in RCA₀. We use \( \{0, 1\}^\omega \) or \( 2^\omega \) to denote the full binary tree. Weak König’s lemma is the following statement: Every infinite subtree of \( 2^\omega \) has an infinite path.

• WKL₀ is defined to be the subsystem of \( \text{Z}_2 \) consisting of RCA₀ plus weak König’s lemma.

• In fact, WKL₀ is strong enough to prove many well known nonconstructive theorems that are extremely important for mathematical practice but not probable in RCA₀.
Theorem 5

Within $\text{RCA}_0$ one can prove that $\text{WKL}_0$ is equivalent to each of the following ordinary mathematical statements:

1. The Heine/Borel covering lemma: Every covering of the closed interval $[0,1]$ by a sequence of open intervals has a finite subcovering.
2. Every covering of a compact metric space by a sequence of open sets has a finite subcovering.
3. The maximum principle: Every continuous real-valued function on $[0,1]$, or on any compact metric space has, or attains, a supremum.
4. Gödel’s completeness theorem: every finite, or countable, set of sentences in the predicate calculus has a countable model.
5. Every countable commutative ring has a prime ideal.
6. The separable Hahn/Banach theorem.
We have seen that WKL\_0 is much stronger than RCA\_0 with respect to mathematical practice. Nevertheless, it can be shown that WKL\_0 is the same strength as RCA\_0 in a proof theoretic sense. Namely, the first order part of WKL\_0 is the same as that of RCA\_0, \( \Sigma^0_1 \)-PA.

Another key conservation result is that WKL\_0 is conservative over the formal system known as PRA or primitive recursive arithmetic, with respect to \( \Pi^0_2 \) sentences. In particular, we can find a primitive recursive function \( f : \omega \to \omega \) such that \( \varphi(m, f(m)) \) holds for all \( m \in \omega \). It means that a large portion of infinitistic mathematical practice is in fact finitistically reducible. Thus we have a significant partial realization of Hilbert’s program of finitistic reductionism.
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Ramsey’s Theorem. The following definitions are made in RCA₀. For any countable \( X \subseteq \omega \) and \( k \in \omega \), let \([X]^k\) be the set of all increasing sequences of length \( k \) of elements of \( X \). In symbols, \( s \in [X]^k \) if and only if \( s \in \omega^k \) and
\[
\forall j < k (s(j) \in X \land \forall i < j (s(i) < s(j))).
\]
By \( \omega \rightarrow (\omega)_l^k \), we mean the assertion that for some \( l \in \omega \) and all \( f : [\omega]^k \rightarrow l \), there exists \( i < l \) and an infinite set \( X \subseteq \omega \) such that \( f(m_1, ..., m_k) = i \) for all \( \langle m_1, ..., m_2 \rangle \in [X]^k \).

It’s easy to show that for each \( k, l \in \omega \), \( \omega \rightarrow (\omega)_l^k \) is provable in ACA₀.

Over RCA₀, ACA₀ is equivalent to \( \omega \rightarrow (\omega)_l^k \) where \( k, l \in \omega \) and \( k \geq 3 \).
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In 1980’s English Logician Seetapun showed that there is an \( \omega \)-model of \( \text{WKL}_0 + \omega \rightarrow (\omega)_l^2 \) in which \( \text{ACA}_0 \) fails.

The existence of an \( \omega \)-model of \( \text{WLK}_0 \) in which \( \omega \rightarrow (\omega)_l^2 \) fails is due to Hirst, 1987.

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Cholak, Jockusch and Slaman showed the following results.
(1). The existence of an \( \omega \)-model of \( \text{RCA}_0 \) which \( \omega \rightarrow (\omega)_2^2 \) fails;
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- \( \Pi^1_1-\text{CA}_0 \) is the subsystems of \( Z_2 \) whose axioms are the basic axioms, the induction axiom, and the comprehension scheme restricted to \( L_2 \)-formulas \( \varphi(n) \) which are \( \Pi^1_1 \). Thus we have the universal closure of

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- There are certain exceptional theorems of ordinary mathematics which can proved in \( \Pi^1_1-\text{CA}_0 \) but cannot be proved in \( \text{ACA}_0 \). The exceptional theorems come from several branches of mathematics including countable algebra, the topology of the real line, countable combinatorics, and classical descriptive set theory.
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Example:

Within ACA\(_0\) we define a countable linear ordering to be a structure \(\langle A, <_A \rangle\), where \(A \subseteq \omega\) and \(<_A \subseteq A \times A\) is an irreflexive linear ordering of \(A\). The countable linear ordering \(\langle A, <_A \rangle\) is called countable well ordering if there is no sequence \(\langle a_n : n \in \omega \rangle\) of elements of \(A\) such that \(a_{n+1} <_A a_n\) for all \(n \in \omega\). Two countable well ordering \(\langle A, <_A \rangle, \langle B, <_B \rangle\) are said to be comparable if they are isomorphic if one of them is isomorphic to a proper initial segment of the order.

The fact that any countable well ordering are comparable turn out to be proved in \(\Pi^1_1\)-CA\(_0\) but not in ACA\(_0\). Thus \(\Pi^1_1\)-CA\(_0\), but not ACA\(_0\), is strong enough to develop a good theory of countable ordinal numbers.
Arithmetical Transfinite Recursion (ATR). Consider an arithmetical formula $\theta(n, X)$ with a free number variable $n$ and a free set variable $X$. Note that $\theta(n, X)$ may also contain parameters. Fixing these parameters, we may view $\theta$ as an “arithmetical operator” $\Theta : P(\omega) \to P(\omega)$, defined by

$$\Theta(X) = \{n \in \omega : \theta(n, X)\}.$$

Now let $\langle A, <_A \rangle$ be any countable well ordering, and consider the set $Y \subseteq \omega$ obtained by transfinitely iterating the operator $\Theta$ along $\langle A, <_A \rangle$. This set $Y$ is defined by the following conditions: $Y \subseteq \omega \times A$ and, for each $a \in A$, $Y_a = \Theta(Y^a)$, where $Y^a = \{m : (m, a) \in Y\}$ and $Y^a = \{(n, b) : n \in Y_a \land b <_A a\}$.

ATR is the axiom scheme asserting that such a set $Y$ exists.
Informally, arithmetical transfinite recursion can be described as the assertion that the Turing jump operator can be iterated along any countable well ordering starting any set.

We define $\text{ATR}_0$ to consist of $\text{ACA}_0$ plus the scheme of arithmetical transfinite recursion. It is easy to see that $\text{ATR}_0$ is a subsystem of $\Pi^1_1$-$\text{CA}_0$. Furthermore, it is a proper subsystem.

$\text{ATR}_0$ is sufficiently strong to accommodate a large portion of mathematical practice beyond $\text{ACA}_0$, including many basic theorems of infinitary combinatorics and classical descriptive theory.
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As a perhaps not unexpected byproduct, we note that these same five systems turn out to correspond to various well known, philosophically motivated programs in foundations of mathematics, as indicated in following table.

<table>
<thead>
<tr>
<th>System</th>
<th>Philosophy</th>
<th>Author(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>RCA₀</td>
<td>Constructivism</td>
<td>Bishop</td>
</tr>
<tr>
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</tr>
<tr>
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<td>Predicativism</td>
<td>Weyl, Feferman</td>
</tr>
<tr>
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<td>Predicative reductionism</td>
<td>Friedman, Simpson</td>
</tr>
<tr>
<td>$\Pi^1_1$-CA₀</td>
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</tr>
</tbody>
</table>
Reverse Mathematics and Other Branches

- Reverse mathematics and higher recursion theory.
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- Reverse mathematics and set theory.
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References


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