An Introduction to Forcing

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Outline

1. Preliminary and Intention
   - Preliminary
   - Intention

2. Forcing and Consistency Proofs
   - Generic Extension
   - Forcing Relation
   - Forcing with Finite Partial Functions
1 Our Logic
- Soundness and Completeness of first-order predicate logic
  \( T \) is consistent if and only if \( T \) has a (countable) model.
- Gödel’s incompleteness results
  We can only hope relative consistency results, e.g.

\[
\text{Con}(\text{ZFC}) \rightarrow \text{Con}(\text{ZFC} + V = L)
\]

2 Our Theory of Sets
- Axioms of ZFC
- Partial orders, boolean algebras, filters, dense sets, chain/antichain, etc.
- Relativization and absoluteness
- Others: \( \Delta \)-system, cardinal arithmetic, etc.
We have found a “model” \( L \) of constructible sets in the ground model and shown that

\[
(ZF + V = L)^L, \quad V = L \rightarrow \text{GCH} \land \text{AC}. \tag{1}
\]

No inner model can be found to make \((ZF + V \neq L)\) true in it as long as ZFC is consistent.

We should extend our ground model \( M \) to be \( M[G] \), the generic extension.
Basic Idea

- Start from $M$, a countable, transitive model of ZFC.
- Design a partial order $\mathbb{P}$ (the set of conditions) in $M$.
- Pick a generic filter $G \subseteq \mathbb{P}$, usually $G \notin M$.
- Make $M[G]$ the smallest transitive model of ZFC containing both $M$ and $G$.
- The truth in $M[G]$ base mainly on the ground model $M$ and the partial order $\mathbb{P}$. 

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**Foundational Theorem of Forcing**

**Theorem (Theorem of Generic Model)**

Given ground model $M$, partial order $\mathbb{P} \in M$, and generic filter $G$, there is a $M[G]$ such that

- $M[G]$ is a transitive model of ZFC;
- $M \subseteq M[G]$ and $G \in M[G]$;
- $M[G]$ is the smallest such model.

**Theorem (Forcing Theorem)**

Under the hypotheses of the previous theorem. Given formula $\varphi(v_1, \ldots, v_n)$ and $\mathbb{P}$-name $\tau_1, \ldots, \tau_n \in M$.

$$\varphi(\tau_1^G, \ldots, \tau_n^G)^{M[G]} \text{ if and only if } \exists p \in G(p \models \varphi(\tau_1, \ldots, \tau_n))^M.$$
Generic Filter

**Definition**

\( G \subseteq \mathcal{P} \) is a **generic filter** if \( G \) is a filter and for each dense \( D \subseteq \mathcal{P} \) such that \( D \in M \), \( G \cap D \neq \emptyset \).

- We can always found a generic filter in a countable ground model.
- We can do forcing from arbitrary partial order \( \mathcal{P} \), but only the following case is nontrivial.

  For each \( p \in \mathcal{P} \), there are \( q \leq p \) and \( r \leq p \) such that \( q \perp r \).
People in $M$ should think about possible extensions, and denote the objects in them by $\mathbb{P}$-names.

**Definition**

$\tau$ is $\mathbb{P}$-name if $\tau$ is a relation, and for all $(\pi, p) \in \tau$, $\pi$ is $\mathbb{P}$-name, $p \in \mathbb{P}$.

- The definition of $\mathbb{P}$ must be considered as inductive.
- $\mathbb{P}$-name is an absolute notion.
- $\mathbb{P}$-names can be ranked.
The Generic Extension $M[G]$

We define the object $\tau^G$ that the name $\tau$ denotes, and $G$ assigns to.

**Definition**

$\tau$ is $\mathbb{P}$-name,

\[
\tau^G = \{ \pi^G \mid (\exists p \in G)(\pi, p) \in \tau \}. \tag{2}
\]

Note that the definition is also inductive. The **generic extension** is defined as,

**Definition**

\[
M[G] = \{ \tau^G \mid \tau \in M^\mathbb{P} \}. \tag{3}
\]

$M[G]$ is transitive if $M$ is.
The Canonical Names

Definition

For each set $x$ in the ground model, we define

$$ \check{x} = \{ (\check{y}, p) \mid y \in x, p \in \mathbb{P} \}. $$

(4)

We claim that $\check{x}^G = x$. Thus $M \subseteq M[G]$.

Definition

$$ \hat{G} = \{ (\hat{p}, p) \mid p \in \mathbb{P} \}. $$

(5)

- $\hat{G}$ has nothing to do with $G$.
- $G = \hat{G}^G \in M[G]$.
- $G$ is the oracle beyond $M$ and finally decides $M[G]$. 
Forcing Relation
Atomic Case

We define the forcing relation \( p \Vdash \varphi(\tau_1, \ldots, \tau_n) \), where \( p \) is a condition, \( \tau_1, \ldots, \tau_n \) are \( \mathbb{P} \)-name.

Definition

1. For atomic formula, we define by induction on \((\text{rank}(\tau_1), \text{rank}(\tau_2))\)

   \begin{itemize}
   \item \( p \Vdash \tau_1 = \tau_2 \) iff \( p \Vdash \tau_1 \subseteq \tau_2 \) and \( p \Vdash \tau_2 \subseteq \tau_1 \),
   \item \( p \Vdash \tau_1 \subseteq \tau_2 \) iff for each \((\pi, r) \in \tau_1\),
     \{q \mid q \leq r \rightarrow q \Vdash \pi \in \tau_1\} is dense below \( p \);
   \item \( p \Vdash \tau_1 \in \tau_2 \) iff \{q \mid \exists(\pi, r) \in \tau_2 (q \leq r \land q \Vdash \pi = \tau_1)\} is dense below \( p \).
   \end{itemize}

The atomic case is absolute.
We continue the definition by induction on the complexity of formula.

**Definition**

1. \( p \models \varphi \land \psi \) if and only if \( p \models \varphi \) and \( p \models \psi \);
2. \( p \models \neg \varphi \) if and only if for each \( q \leq p \), \( q \not\models \varphi \);
3. \( p \models \exists x \varphi(x) \) if and only if
   \[ \{ q \mid \exists \pi ( \pi \text{ is } \mathbb{P}\text{-name} \land q \models \varphi(\pi) ) \} \text{ is dense below } p. \]

- Forcing relation is not absolute generally.
- The forcing relation is the “logic” of the people living in \( M \). It decides the outline of every possible \( M[G] \).
Some Additional Property of Forcing Relation

Theorem

- If $q \leq p$, then $p \forces \varphi$ implies $q \forces \varphi$.
- $\{p \mid p \forces \varphi \lor p \forces \neg \varphi\}$ is dense.
- No $p \in \mathbb{P}$ forces both $\varphi$ and $\neg \varphi$. 
Proof of the Forcing Theorem

We prove that

$$\varphi(\tau_1^G, \ldots, \tau_n^G) \text{ iff } \exists p \in G (p \Vdash \varphi(\tau_1, \ldots, \tau_n))^M$$

- Atomic case
- Boolean and quantifier case
Finish the Proof of the Generic Model Theorem

Lemma

\[ M[G] \models ZFC. \]

Proof.

- Extensionality: \( M[G] \) is transitive.
- Foundation: holds in each \( \in \) model.
- For those axioms that asserts existence of sets, we should design appropriate names.

Lemma

If \( N \) is a transitive model of ZFC and that \( M \subseteq N, G \in N \), then \( M[G] \subseteq N \).
The crucial trick is to design the partial order. Here we give a simple example.

**Definition (Finite partial functions)**

\[ Fn(I, J) = \{ p : |p| < \omega \land p \text{ is a function} \land \text{dom } p \subseteq I \land \text{ran } p \subseteq J \}. \]  

The order on \( Fn(I, J) \) is defined as

\[ p \leq q \iff p \supseteq q. \]
Some Example

- Forcing with $Fn(\omega, \omega_1)$

$$\bigcup G \text{ is a total function mapping } \omega \text{ onto } \omega_1? \quad (8)$$

- Forcing with $Fn(\kappa \times \omega, 2)$
  - $f = \bigcup G : \kappa \times \omega \mapsto 2$ is total.
  - For $\alpha < \kappa$, Letting
    $$f_\alpha : \omega \mapsto 2, \text{ such that } f_\alpha(n) = f(\alpha, n). \quad (9)$$
  - $\langle f_\alpha : \alpha < \kappa \rangle$ is an one-to-one sequence mapping $\kappa$ into $2^\omega$ in $M[G]$. 
Preserving Cardinals

**Theorem**

\( \mathbb{P} \in M. \) If \( (\mathbb{P} \text{ is c.c.c.)}^M \), then for each generic \( G \) of \( \mathbb{P} \) over \( M \) and ordinal \( \alpha \in M \),

\[ (\alpha \text{ is a cardinal})^M \leftrightarrow (\alpha \text{ is a cardinal})^{M[G]} . \]

**Lemma**

\( Fn(\kappa \times \omega, 2) \) is c.c.c..

Use the \( \Delta \)-system theorem.
Further Discussion

Theorem

For each $\kappa$ with $\text{cf } \kappa > \omega$.

$$\text{Con}(ZFC) \rightarrow \text{Con}(ZFC + 2^\omega = \kappa).$$

- We have shown that $\text{Con}(ZFC) \rightarrow \text{Con}(ZFC + 2^\omega > \kappa)$ for each $\kappa$.
- Forcing with a ground model of $ZFC + \text{GCH}$. 
The generic extension $M[G]$ is built from the ground model $M$, the partial order $\mathbb{P} \in M$, and the generic filter $G$ (usually not in $M$).

The general truth in $M[G]$ is already described by the forcing relation in $M$, so is decided mainly by $\mathbb{P}$ and $M$.

Outlook

- Proper Forcing.
- $\mathbb{P}_{max}$ Forcing.
For Further Reading

T. Jech.  
*Set Theory.*  

K. Kunen.  
*Set Theory: An Introduction to Independence Proofs.*  
Elsevier Science Publisher B.V., 1983.

P. J. Cohen.  
Independence Results in Set Theory.  

T. Y. Chow.  
A beginner’s guide to forcing.  