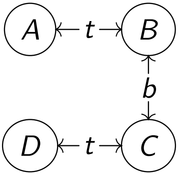
Representing Imperfect Information of Procedures with Hyper Models

Yanjing Wang
Department of Philosophy, Peking University

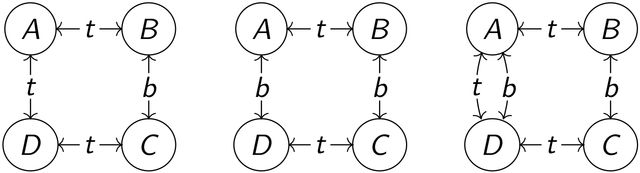
January 10th 2015
ICLA 2015, IIT-Bombay

Four cities and an outdated map



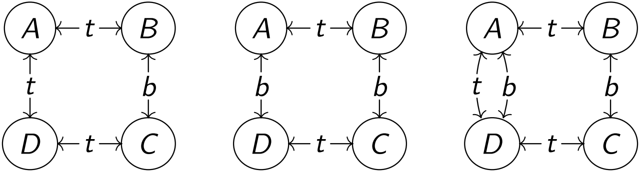
"A and D are now connected directly by public transportation."

- I know that C is now reachable from A via D.
- I think it may be possible to go to C from A by taking train only.



Standard epistemic semantics and updates

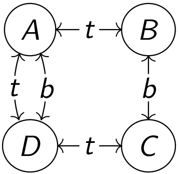
- I know that φ if it is true on *all* the possibilities that I can reasonably think of, given the information that I have.
- φ is possible to me if it is true on *some* possibility.
- New information helps to eliminate some possibilities.



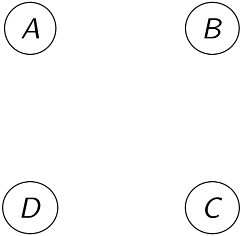
"*A* is connected with *D* by both bus and train."

Standard epistemic semantics, and eliminative updates

- I know that φ if it is true on *all* the possibilities that I can reasonably think of, given the information that I have.
- φ is possible to me if it is true on *some* possibility.
- New information helps to eliminate some possibilities.



Ignorance requires a “big brain”



4 cities and 4 transportation methods: there are more than $\mathbf{10^{14}}$ possible maps!

We definitely do not store them explicitly in our brain to retrieve knowledge. Is there a more cognitively friendly way to give the semantics?

Two questions

- Is there a compact alternative model of epistemic logic of procedures without explicitly representing all the possibilities?
- Is there a way to semantically incorporate (imperfect) procedural information in an incremental fashion from scratch?
- Yes! (but it is not perfect and we need to pay the price)

Main idea

Using the ideas behind 3-valued **abstraction** of model checking [Larsen & Thomsen 88][Bruns & Godefroid 99] [Dams, Gerth & Grumberg 97].

3-valued model abstraction

From 2-valued big models to 3-valued small models while preserving formulas with a definite truth value.

Key ingredients in model abstraction:

- Abstract models with over- and under-approximations of the transitions that represent collections of concrete models.
- Abstraction-refinement relation between such models.
- A 3-valued semantics for modal logic using imperfect information of the transitions.

Towards a 2-valued epistemic logic: interpret truth of φ as knowing that φ, falsity of φ as knowing that $\neg\varphi$, and the third value as "I don't know".

Imperfect information of procedures

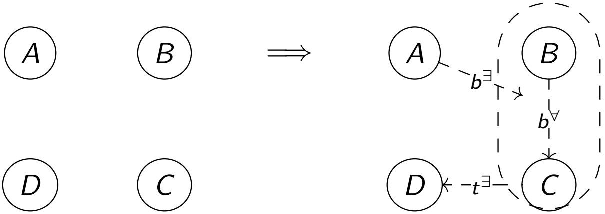
A piece of imperfect information of procedure is treated as a Hoare-like triple: $\langle \varphi, X, \psi \rangle$ where X is one of $\pi^{\exists}$ or $\pi^{\forall}$ for some specification π of a procedure.

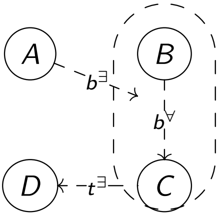
- $\langle \varphi,\ \pi^{\exists},\ \psi \rangle$ says that if φ holds then there *exists* an execution of π which can make ψ true, e.g., "One of these two buses will get you to the university from home."
- $\langle \varphi,\ \pi^{\forall},\ \psi \rangle$ says that if φ holds then *all* the executions of π will make sure ψ, e.g., "All the buses departing here will get you to the university."

A simple example from scratch

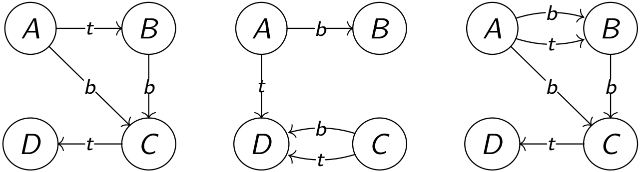
Let p_y only hold on $y \in \{A, B, C, D\}$. Here are three pieces of information:

$$\langle p_A,\ b^{\exists},\ p_B \vee p_C\rangle, \quad \langle p_C,\ t^{\exists},\ p_D\rangle, \quad \langle p_B,\ b^{\forall},\ p_C\rangle.$$





We hope it can represent the possibilities like below which are consistent with the given information.



The agent should know: *there is a bus from A to either B or C, and if it reaches C then D can be reached by a train, otherwise take any bus (if available) from B to get C first in order to reach D finally.*

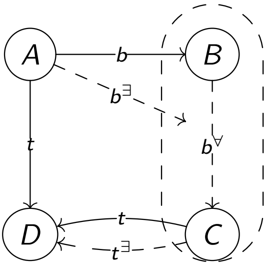
Simple hyper model

Definition (Simple hyper model)

A simple hyper model is a tuple $(S, \rightarrow, \rightarrow_\exists, \rightarrow_\forall, V)$ where:

- $(S, \rightarrow, V)$ is a Kripke model w.r.t. the label set $\mathbf{\Sigma}$.
- $\rightarrow_\exists \subseteq S \times \mathbf{\Sigma} \times 2^S$ is a labelled binary relation from a state to a set of states.
- $\rightarrow_\forall \subseteq S \times \mathbf{\Sigma} \times 2^S$ is a labelled binary relation from a state to a set of states.
- for all $s \in S, T \subseteq S$: $s \xrightarrow{a}_\exists T$ implies that there exists $t \in T$ such that $s \xrightarrow{a} t$.
- for all $s \in S, T \subseteq S$: $s \xrightarrow{a}_\forall T$ implies that for all $t \in S$: $s \xrightarrow{a} t$ implies $t \in T$.

"Hyper" comes from the hyper transitions in [Shoham & Grumberg 08].



where:

- $S = \{A, B, C, D\}$,
- $\rightarrow = \{(A, b, B), (A, t, D), (C, t, D)\}$,
- $\rightarrow_\exists = \{(A, b, \{B, C\}), (C, t, \{D\})\}$,
- $\rightarrow_\forall = \{(B, b, \{C\})\}$,
- for all $s, v \in \{A, B, C, D\}$, $p_s \in V(v)$ iff $s = v$.

Definition (Epistemic action language EAL)

Given a set of *propositional variables* $\mathbf{P}$, a set of *actions* $\mathbf{\Sigma}$, the *formulas* of EAL are given by:

$$\varphi \quad ::= \quad \top \mid p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K\varphi \mid \langle a \rangle \varphi$$

where $p \in \mathbf{P}$ and $a \in \mathbf{\Sigma}$.

As usual, we define $\bot$, $\varphi \vee \psi$, $\varphi \rightarrow \psi$, $\hat{K}\varphi$ and $[a]\varphi$ as the abbreviations of $\neg\top$, $\neg(\neg\varphi \wedge \neg\psi)$, $\neg\varphi \vee \psi$, $\neg K \neg\varphi$ and $\neg\langle a \rangle \neg\varphi$ respectively.

The semantics

The semantics is given w.r.t. a mode $x \in \{0,\ \Box,\ \Diamond\}$:

$$
\begin{array}{rcl}
\mathcal{M}, s \models \varphi & \Leftrightarrow & \mathcal{M}, s \models_0 \varphi \\
\mathcal{M}, s \models_x p & \Leftrightarrow & p \in V(s) \\
\mathcal{M}, s \models_x \varphi \wedge \psi & \Leftrightarrow & \mathcal{M}, s \models_x \varphi \text{ and } \mathcal{M}, s \models_x \psi \\
\mathcal{M}, s \models_x K\varphi & \Leftrightarrow & \mathcal{M}, s \models_\Box \varphi \\
\mathcal{M}, s \models_x \neg\varphi & \Leftrightarrow & \begin{cases} \mathcal{M}, s \not\models_0 \varphi & \text{IF } x = 0 \\ \mathcal{M}, s \not\models_\Diamond \varphi & \text{IF } x = \Box \\ \mathcal{M}, s \not\models_\Box \varphi & \text{IF } x = \Diamond \end{cases} \\
\mathcal{M}, s \models_x \langle a \rangle \varphi & \Leftrightarrow & \begin{cases} \exists t \in S : s \xrightarrow{a} t \text{ and } \mathcal{M}, t \models_0 \varphi \\ \exists T \subseteq S : s \xrightarrow{a}_\exists T \text{ and } \forall t \in T : \mathcal{M}, t \models_\Box \varphi \\ \forall T \subseteq S : s \xrightarrow{a}_\forall T \text{ implies } \exists t \in T : \mathcal{M}, t \models_\Diamond \varphi \end{cases}
\end{array}
$$

φ is valid in $\mathcal{M}$ ($\mathcal{M} \models \varphi$) if for any $s \in S_{\mathcal{M}}$: $\mathcal{M}, s \models \varphi$. φ is valid if for any $\mathcal{M}$: $\mathcal{M} \models \varphi$.

Three modes

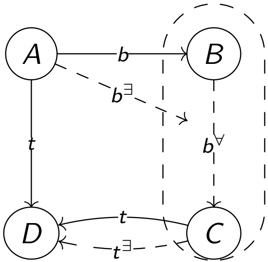
0 marks the factual mode: evaluating formulas outside the scope of any knowledge operator, while $\Box$ and $\Diamond$ denote the *knowledge modes* with the following intentions:

- $\models_{\Box} \varphi$: the agent thinks that φ is *necessarily* true, i.e., φ is true in *all* the actual situations consistent with the procedural information that he has.
- $\models_{\Diamond} \varphi$: the agent thinks that φ is *possibly* true, i.e., φ is true in *some* of the actual situations consistent with the procedural information that he has.

The semantics

$$
\begin{array}{rcl}
\mathcal{M}, s \models_x \varphi \vee \psi & \Leftrightarrow & \mathcal{M}, s \models_x \varphi \text{ or } \mathcal{M}, s \models_x \psi \\
\mathcal{M}, s \models_x \varphi \rightarrow \psi & \Leftrightarrow & \begin{cases} \mathcal{M}, s \models_0 \varphi \text{ implies } \mathcal{M}, s \models_0 \psi & x = 0 \\ \mathcal{M}, s \models_\Diamond \varphi \text{ implies } \mathcal{M}, s \models_\Box \psi & x = \Box \\ \mathcal{M}, s \models_\Box \varphi \text{ implies } \mathcal{M}, s \models_\Diamond \psi & x = \Diamond \end{cases} \\
\mathcal{M}, s \models_x \hat{K}\psi & \Leftrightarrow & \mathcal{M}, s \models_\Diamond \psi \\
\mathcal{M}, s \models_x [a]\varphi & \Leftrightarrow & \begin{cases} \forall t : s \xrightarrow{a} t \text{ implies } \mathcal{M}, t \models_0 \varphi \\ \exists T \subseteq S : s \xrightarrow{a}_\forall T \text{ and } \forall t \in T : \mathcal{M}, t \models_\Box \varphi \\ \forall T \subseteq S : s \xrightarrow{a}_\exists T \text{ implies } \exists t \in T : \mathcal{M}, t \models_\Diamond \varphi \end{cases}
\end{array}
$$

The semantics



$\mathcal{M} \vDash p_A \rightarrow \langle b \rangle (p_B \vee p_C)$
$\mathcal{M} \vDash p_B \rightarrow [b] p_C$
$\mathcal{M} \vDash p_C \rightarrow \langle t \rangle p_D$
$\mathcal{M}, A \vDash \langle t \rangle p_D \wedge \neg K \langle t \rangle p_D \wedge \neg K \neg \langle t \rangle p_D$
$\mathcal{M}, B \vDash [b] \neg p_D \wedge K [b] \neg p_D$
$\mathcal{M}, C \vDash \langle t \rangle p_D \wedge K \langle t \rangle p_D$
$\mathcal{M}, A \vDash K \langle b \rangle ((p_C \rightarrow \langle t \rangle p_D) \wedge (p_B \rightarrow [b] p_C)$

$$\begin{aligned}
& \mathcal{M}, A \vDash \neg K \langle t \rangle p_D \wedge \neg K \neg \langle t \rangle p_D \\
\Longleftrightarrow\ & \mathcal{M}, A \vDash_0 \neg K \langle t \rangle p_D \text{ and } \mathcal{M}, A \vDash_0 \neg K \neg \langle t \rangle p_D \\
\Longleftrightarrow\ & \mathcal{M}, A \nvDash_0 K \langle t \rangle p_D \text{ and } \mathcal{M}, A \nvDash_0 K \neg \langle t \rangle p_D \\
\Longleftrightarrow\ & \mathcal{M}, A \nvDash_\Box \langle t \rangle p_D \text{ and } \mathcal{M}, A \nvDash_\Box \neg \langle t \rangle p_D \\
\Longleftrightarrow\ & (\text{It is not the case } (\exists T \subseteq S : A \xrightarrow{t}_\exists T \text{ and } \forall v \in T : \mathcal{M}, v \vDash_\Box p_D)) \\
& \text{ and } \mathcal{M}, A \vDash_\Diamond \langle t \rangle p_D \\
\Longleftrightarrow\ & \forall T \subseteq S : A \xrightarrow{t}_\forall T \text{ implies } \exists v \in T : \mathcal{M}, v \vDash_\Diamond p_D
\end{aligned}$$

Is it a reasonable epistemic semantics?

Lemma

For all the pointed simple hyper model $\mathcal{M}, s$, any EAL formula φ, the following two hold:

1. $\mathcal{M}, s \vDash_{\Box} \varphi$ *implies* $\mathcal{M}, s \vDash_0 \varphi$
2. $\mathcal{M}, s \vDash_0 \varphi$ *implies* $\mathcal{M}, s \vDash_{\Diamond} \varphi$

Therefore $\mathcal{M}, s \vDash_{\Box} \varphi$ *implies* $\mathcal{M}, s \vDash_{\Diamond} \varphi$.

Theorem

The following **S5** *axiom schemas are valid:* $K(\varphi \to \psi) \to (K\varphi \to K\psi)$ $\quad K\varphi \to \varphi \quad K\varphi \to KK\varphi \quad \neg K\varphi \to K\neg K\varphi$. *However, the rule of Necessitation* ($\vDash \varphi$ *implies* $\vDash K\varphi$) *is* **not** *valid.*

The full language

Definition (Epistemic PDL)

The *formulas* φ of our EPDL language are given by:

$$\begin{array}{lcl} \varphi & ::= & \top \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi \mid \langle\pi\rangle\varphi \\ \pi & ::= & a \mid \pi;\pi \mid \pi+\pi \mid \pi^* \end{array}$$

Hyper models

Definition (Hyper model)

An hyper model is a tuple $(S, \rightarrow, \rightarrow_\exists, \rightarrow_\forall, V)$ where:

- $(S, \rightarrow, V)$ is a Kripke model
- $\rightarrow_\exists \subseteq 2^S \times \Pi_{\mathbf{\Sigma}} \times 2^S$ is a labelled binary relation from a set of states to a set of states.
- $\rightarrow_\forall \subseteq 2^S \times \Pi_{\mathbf{\Sigma}} \times 2^S$ is a labelled binary relation from a set of states to a set of states.
- for all $T, T' \subseteq S$: $T \xrightarrow{\pi}_\exists T'$ implies that for all $t \in T$ there exists $w \in \pi$ and $t' \in T'$ such that $t \xrightarrow{w} t'$.
- for all $T, T' \subseteq S$: $T \xrightarrow{\pi}_\forall T$ implies that for all $t \in T$ all $w \in \pi$: $t \xrightarrow{w} t'$ implies $t' \in T'$.

where $\Pi_{\mathbf{\Sigma}}$ is the set of regular expressions.
Previous lemma and theorem also hold here w.r.t. a rather complicated semantics.

Representation of a class of models

Is each hyper modal really a compact representation of a class of Kripke models?

Let $Unf(\mathcal{M})$ be the collection of Kripke models with the same state space that are consistent with the imperfect information in $\mathcal{M}$. From the "$\Box$ to $\Diamond$" lemma:

Proposition

For every PDL formula φ and every s in any hyper model $\mathcal{M}$:

- *if $\mathcal{M}, s \vDash K\varphi$ then $\mathcal{N}, s \vDash \varphi$ for every $\mathcal{N} \in Unf(\mathcal{M})$*
- *if $\mathcal{N}, s \vDash \varphi$ for some $\mathcal{N} \in Unf(\mathcal{M})$ then $\mathcal{M}, s \vDash \hat{K}\varphi$*

Representation of a class of models

Observation

For some classes of Kripke models, there is no corresponding hybrid model which can capture all non-trivial knowledge w.r.t. the class.

As an example, the set of two Kripke models $s \xrightarrow{a} t$ and $s \xrightarrow{b} t$ is not representable by any hyper model.

Let $\varphi = (\langle a\rangle\top \wedge \neg\langle b\rangle\top) \vee (\langle b\rangle\top \wedge \neg\langle a\rangle\top)$. φ is true on both models. However, no matter how the $\rightarrow_\forall$ and $\rightarrow_\exists$ transitions are chosen, we cannot make sure $K\varphi$ holds on s since $K(\chi \vee \psi) \rightarrow K\chi \vee K\psi$ is valid due to our semantics. $K(\chi \vee \psi)$ should be interpreted as *knowing whether* χ or ψ. It is hard to capture the inter-dependency of the procedural information.

Another catch

Proposition

For every PDL formula φ and every s in any hyper model $\mathcal{M}$:

- *if $\mathcal{M}, s \vDash K\varphi$ then $\mathcal{N}, s \vDash \varphi$ for every $\mathcal{N} \in Unf(\mathcal{M})$*
- *if $\mathcal{N}, s \vDash \varphi$ for some $\mathcal{N} \in Unf(\mathcal{M})$ then $\mathcal{M}, s \vDash \hat{K}\varphi$*

We should try to retrieve as much as the information that is already encoded by the hyper model!

- $(s \xrightarrow{a}_\forall T_1$ and $s \xrightarrow{a}_\forall T_2)$ implies $s \xrightarrow{a}_\forall T_1 \cap T_2$
- $(s \xrightarrow{a}_\forall T_1$ and $s \xrightarrow{a}_\exists T_2)$ implies $s \xrightarrow{a}_\exists T_1 \cap T_2$
- for each s each a, there exists $T \subseteq S$ such that $s \xrightarrow{a}_\forall T$
- $K[a]\varphi \wedge K[a]\psi \rightarrow K[a](\varphi \wedge \psi)$
- $K[a]\varphi \wedge K\langle a\rangle\psi \rightarrow K\langle a\rangle(\varphi \wedge \psi)$
- $K[a]\top$

Conclusion

We developed a **2-valued Epistemic PDL** framework based on **abstract** models **without** using explicit epistemic possibility relations as in S5 models.

- Each abstract model assembles a set of concrete possibilities in a very compact way.
- The semantics of our language is defined on the abstract model directly (no need to unpack the concrete models).
- The logic is still 2-valued.
- The models can be built step by step from scratch by incorporating the imperfect information.
- All the S5 axioms are valid.
- We also pay some price and the logic is very weak so far.

This may be the beginning of an interesting story and a lot of efforts are to be made (measure of informativeness, MC, axiomatization...)!

The background research program: Beyond knowing that

Knowledge is not only expressed in terms of "knowing that":

- I *know whether* the claim is true.
- I *know how* to go to Chennai.
- I *know what* your password is.
- I *know who* proved this theorem.
- ...

BKT: to study the logical properties of these new operators, which may lead to:

- interesting non-normal 'modal' operators and axioms
- discovery of new decidable ("guarded") fragments of first-order modal logic
- cognitively /computationally friendly knowledge representations
- insights for questions in philosophy and linguistics

Beyond "knowing that": join us!

ZXCV at Greenlife

Thank you very much for your attention!

Grumberg, O. (2010).
2-valued and 3-valued abstraction-refinement in model checking.
In *Logics and Languages for Reliability and Security*, pages 105–128.

Shoham, S. and Grumberg, O. (2008).
3-valued abstraction: More precision at less cost.
Information and Computation, 206(11):1313–1333.