Philosophical ontologies

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Philosophical ontologies

Plato 380BC

Mathematics views its most cherished answers only as springboards to deeper questions. (Barry Mazur 2003)

Inquiring into things we do not know will make us better, and braver, and less lazy than believing we cannot discover anything we do not already know. (Plato)

Plato (and the history of mathematics) teaches us:

Foundations. There can never be final logical, or working, or conceptual foundations for mathematics.

1. But mathematics has to seek these.

Ontology. Mathematical objects are not entirely real.

1. Intermediate between “what comes to be and passes away” and “what is always and in every way.”

For a Platonist the Forms are yet more real and still more fundamental to explaining the scheme of things than the objects of mathematics. (Myles Burnyeat)

No beautiful thing in the visible world is in every way beautiful. The form of beauty is.

No beautiful thing in the visible world is beautiful forever.

The Form of beauty itself exists “always and in every way.”

Forms=Ideas are perfectly real. Philosophers *long to* know them.

Plato is an idealist – but a dialectical idealist.

He believes *the* Ideas never change.

He believes *our* ideas constantly change and develop as we use them.

Plato knew the advanced mathematics, and the advanced mathematicians, of his time.

Plato knew Theaetetus and Eudoxus.

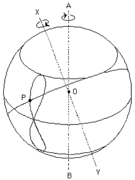
Theaetetus developed a theory of irrational lengths, and of regular solids.

Possibly first Greek geometer to know octahedron and icosahedron. (Hard to believe of octahedron, which occurs in crystals.)

Proved these are the only regular solids.

Eudoxus gave the theory of proportions that Dedekind compared to his own theory of irrational numbers.

And studied the motion of the planets.



Theaetetus’ and Eudoxus’ work later became Euclid Books X–XIII.

Plato knew mathematicians create new concepts and goals as well as theorems.

He knew they abandon many along the way.

And they erase the creative process from their polished proofs.

This is why Socrates says in *The Republic*:

> *students of geometry and reckoning and such subjects first hypothesize the odd and the even and the various figures and three kinds of angles and other things like these in each inquiry, regard them as known, and, making these their hypotheses, do not explain them to themselves or others, as if they are obvious to everybody.*

Notice these hypotheses are not statements, they are concepts.

For Plato, hypotheses (ὑποθέσεις) can be assumptions, or can be concepts.

Sometimes he calls a goal an hypothesis. For example, he writes that making the citizens happy is the hypothesis of the city state.

In short, an hypothesis for Plato is any starting point for an inquiry.

Failed hypotheses get replaced. Others work well enough but then fade from interest. Successful ones produce their own successors.

We will come back to this in connection with dialectic.

Plato also knew the difference between a goal and an achievement.

In *The Republic* Socrates says:

> *[The geometers] speak quite laughably, and that by compulsion, as if they are acting and all their accounts are for the sake of action. They talk of "squaring", "applying", "adding" and the like, whereas the entire subject is pursued for the sake of knowledge of what always is.*

Geometry is *meant to be* knowledge of changeless things.

Yet the geometers cannot avoid talking about changing the diagrams.

The young man Glaucon (actually Plato's brother) is a character in *The Republic*. He tries to agree with Socrates. But he gets a lot wrong.

Glaucon tries to agree by saying "Geometry is knowledge of what always is." But Socrates laughs at him.

Socrates answers that geometry is

> *a powerful device ... pulling the soul towards truth and philosophic thought, by directing upwards what we now wrongly direct downwards.*

In *The Republic* Socrates calls geometry a *science*, which should mean ideal truth.

But he also talked about the science of boxing, and the science of house building.

He eventually says all these were mistakes, “based on habit.”

Geometry is closer to truth than boxing is, or house building. So he says geometry needs “some other name connoting more clarity than opinion, and less than science.”

The problem is that geometric proofs draw conclusions from their hypotheses, without changing the hypotheses in the process.

The hypotheses are treated as fixed, and not as “genuine hypotheses, or stepping stones” on the way to “the unhypothetical first principle of everything.”

Only dialectic has genuine hypotheses.

Plato knows the geometer's ideal is never achieved. The geometer's hypotheses are not truly fixed.

This is why they are *compelled* to work in laughable ways. This is why geometry cannot find "what truly is."

Socrates says geometers "to some extent" grasp what is:

> *They dream about what is, unable to get a waking view of it as long as they use hypotheses that they leave unaltered and that they cannot give an account of.*

He says "dreaming" is mistaking what resembles a thing for that thing.

for Plato's character Socrates, mathematical objects resemble "what is" without being "what is."

This is a valuable perspective on the actual practice of mathematics.

Even the great Platonist philosopher Burnyeat calls the Pythagorean Theorem "an eternal, context-invariant truth."

On my view, Plato offers general reasons to doubt this. The theorem rested on unexamined hypotheses and so could change.

Whether or not Plato thought it *would* ever change its meaning and its status, it *did*.

Geometers before 1800 took the Pythagorean Theorem as the plain truth.

Today it is merely an equivalent for the parallel postulate. It gives one of many geometries. And it only approximates the space we live in.

Riemann's 1854 dissertation "On the hypotheses that lie at the base of geometry" changed mathematical geometry even more than the non-Euclidean geometries did.

And then General Relativity applied that change to physical space – or rather, spacetime.

Quantum gravity has demanded yet further new ideas of geometry.

Many other problems are doing the same.

The pace of change in geometry is only rising.

The success of set theory in clarifying mathematics led to the creation of category theory.

Category theory claims to put set theory into a new perspective.

Category theory has led to higher category theory, and homotopy type theory. . . .

Even within ZFC itself, Gödel's, and Cohen's success on the continuum hypothesis makes set theorists ask if they need new axioms.

To believe a foundation for geometry or mathematics could be fixed for all time is to disagree with Plato.

And it is to disagree with the history of mathematics.

Philosophical ontologies

The philosophy of mathematicians in Hilbert's Göttingen

No one shall drive us out of the paradise which Cantor has created for us. (Hilbert 1926)

David Hilbert, after thirty years of high creative achievement on the frontiers of mathematics, walked into the blind alley of reductionism. . . . aimed to reduce the whole of mathematics to a collection of formal statements using a finite alphabet of symbols. . . .

Hilbert then proposed to solve the problems of mathematics by finding a general decision-process that could decide, given any formal statement composed of mathematical symbols, whether that statement was true or false. . . . He dreamed of solving [this] problem and thereby solving as corollaries all the famous unsolved problems of mathematics. (Freeman Dyson)

Saunders Mac Lane showed Dyson was wrong in several ways.

> *I was a student of Mathematical logic in Göttingen in 1931–33, just after the publication of the famous 1931 paper by Gödel. Hence I venture to reply.*
>
> *Hilbert himself called this 'metamathematics.' He used this for a specific limited purpose, to show mathematics consistent. Without this reduction, no Gödel's theorem, no definition of computability, no Turing machine, and hence no computers.*
>
> *Dyson simply does not understand reductionism and the deep purposes it can serve.*

But, much of Hilbert's wording in *On the Infinite* and later seems to agree with Dyson.

As many Hilbert scholars have said, Hilbert develops a “finite standpoint” (*finit Standpunkt*) or “finite position” (*finit Einstellung*) that he finds philosophically irrefutable.

This standpoint will show that all the finite statements provable in Cantor’s infinitary set theory are true – even if you do not believe infinitary statements even mean anything.

Hilbert himself believed in all of Cantor’s set theory.

Hilbert: the single formula contains infinitely many propositions

$$1^2 + 2^2 + 3^2 + \cdots + n^2 \ = \ \frac{1}{6}n(n+1)(2n+1)$$

Finitary reasoning (Δ_0 induction) in our metatheory lets us prove this for each numeral n.

Hilbert:

> *The deepest insight into the nature of the infinite [comes from] a discipline which comes closer to a general philosophical way of thinking and which was designed to cast new light on the whole complex of questions about the infinite. This discipline, created by George Cantor, is set theory.*

Hilbert does not conceive of coding such as Gödel numbering

> *For logical deduction to be certain [the symbols and sequences of symbols must be given directly] This is the basic philosophy which I find necessary, not just for mathematics, but for all scientific thinking, understanding, and communicating.*
>
> *The subject matter of mathematics is, in accordance with this theory, the concrete symbols themselves whose structure is immediately clear and recognizable.*

Hilbert

> *Mathematics surely does not consist solely of numerical equations and surely cannot be reduced to them alone. Still one could argue that mathematics is an apparatus which, when applied to integers, always yields correct numerical equations.*

“From our finite standpoint (*finit Standpunkt*)” an unbounded existential statement is “only a partial statement. . . part of a more determinate [bounded] statement.”

So we “from our finite position” cannot apply excluded middle to unbounded existential statements. We cannot negate *part of* a statement that we do not even have in full. (Brouwer’s influence.)

In the metatheory, Con(PA) is one formula containing infinitely many propositions

$$\text{String } \sigma \text{ is not a proof of } \mathsf{0} = \mathsf{1} \text{ in PA}$$

So Hilbert wants a constructive procedure that takes arbitrary strings as input, and shows each one is not a proof of $\mathsf{0} = \mathsf{1}$ in PA.

Then in 1928 Hilbert and Wilhelm Ackermann asked for more. The second half of Dyson's complaint.

The *Decision problem* (or, *Entscheidungsproblem*) for PA.

A *decision routine* for PA is a constructive procedure that takes as input one sentence ϕ of PA and an arbitrary string of symbols σ in PA, and decides whether or not σ is a PA proof of ϕ.

Suppose PA is complete, so that for every sentence ϕ it does prove either ϕ or not-ϕ.

Then a decision routine would give a mechanical routine to take any sentence of PA (say, the Goldbach conjecture, or twin primes conjecture) and either prove it or refute it.

Hilbert asked if the same could be done for every first order theory in place of PA>

Notably first order set theory.

This would give a routine for all the theorems of mathematics.

Gödel, and then Alan Turing and Alonzo Church showed none of this was possible, even for pure first order logic, let alone PA or set theory.

I also want to point out that Hilbert absolutely did not think a decision routine for set theory would actually settle all the problems of mathematics.

Twenty five years earlier he had found a routine taking any degree polynomial in any number of variable as input, and calculating a (finite) complete set of invariants for it.

He knew that routine was infeasible for one variable and say, degree 6.

As to Hilbert's famous phrase "In mathematics there is nothing we will not know." *In Mathematik gibt es kein Ignorabimus*.

Every mathematical problem can be solved.

Not mechanically, not for arbitrary first order sentences.

He meant: the mathematical questions we ask, we can answer.

Like Plato, Hilbert wanted us to be "better, braver, and less lazy" than if we believe some things just cannot be known.

Hilbert not only had his own philosophy.

He had two philosophers hired at Göttingen, the first was his friend Edmund Husserl.

Plus Hilbert's student and protegé Hermann Weyl was strongly involved with literature and philosophy.

none of these were interested in any ontological minimalism.

Husserl studied with Weierstrass and Kronecker, and wrote his dissertation on calculus of variations.

Hilbert got him hired in philosophy at Göttingen, though the other philosophers were against him.

Husserl did not stay long, thogh he continued visiting Hilbert often.

He was replaced by his friend Moritz Geiger.

Saunders Mac Lane studied with Geiger and was influenced by his philosophy, inclduing his idea that *mathematical forms* exist though they are not physical.

Geiger was interested in both:‘

> *reality (Wirklichkeit) as defined in science, and reality as metaphysics strives to know it.*

He agrees with Kant in part:

> *The sciences advance by secure methods while each metaphysician begins anew, yet we inevitably seek metaphysical clarity and unity, while any attempt to take the assumptions of science as metaphysical absolutes leads to contradiction.*

Geiger distinguishes two equally important attitudes: *naturalistic* and *immediate*.

The naturalistic attitude is common in physical science, and takes no account of people as observers.

> *Psychic and physical are in contradictory opposition for the naturalistic attitude. What is not physical is psychic, and what is not psychic is physical-. . . . The physical is real in space and time, 'objective'; the psychic is non-spatial and 'merely' subjective.*

The immediate attitude is more common in ordinary life.

The immediate attitude takes people into account as observers.

It takes the physical and psychic as real, plus realities that are not physical or psychic, such as poems, or governments.

> *The naturalistic attitude knows only psychic and physical forms (Gebilde). If Mathematics were a science in the naturalistic attitude, it would have to be either a science of physical objects, thus a kind of applied physics, or a science of psychic objects, thus a kind of applied psychology. Yet Mathematics is neither the one nor the other.*

At the same time, Mac Lane was living in Hermann Weyl's house.

He helped Weyl practice English, and helped him translate Weyl's book *Philosophy of Mathematics and Natural Science* into English.

That version never appeared in print. A much later version did.

Weyl was very literary, and quotes many philosophers.

He quotes Heraclitus and Euclid in Greek.

He cites Galileo, Leibniz, Kant, Fichte, Schelling, and Hegel. . . .

Mac Lane was obviously influenced by Weyl’s breadth of mathematics, and his clear essay style (though Weyl’s books are not clear).

Weyl’s short book gives concise philosophic discussion of non-Euclidean geometry, topology, transformation groups in geometry, plus relativity and quantum theory.

Weyl like Geiger spoke of mathematical *forms* with a different order of being than actual things:

> *To the Greeks we owe the recognition that the structure of space, manifest in the relations between spatial forms (Gebilde) and their lawful dependence on one another, is something completely rational. This is unlike the case of an actual particular where we must ever build from new input of intuition.*

He says modern pure mathematics is

> *a purely intellectual mathematics freed from all intuition, a pure theory of forms (Formenlehre) dealing with neither quanta nor their images the numbers, but intellectual objects which may correspond to actual objects or their relations but need not.*

He quotes Husserl that “without this viewpoint [. . .] one cannot speak of understanding the mathematical method.”

Weyl was attracted to Brouwer's intuitionism for the rest of his life (recall Brouwer claimed Weyl did not understand him).

But finally he had to agree with Hilbert that mathematicians cannot leave Cantor's paradise:

> *Mathematics with Brouwer achieves the highest intuitive clarity. He is able to develop the beginnings of analysis more naturally, and in closer contact with intuition, than before. But one cannot deny that, in progressing to higher and more general theories, the unavailability of the simple axioms of classical logic finally leads to nearly insupportable difficulties.*

Philosophical ontologies

Fictionalism, modalism, and Cantor’s freedom of mathematics

All the Second Philosopher’s impulses are methodological, just the thing to generate good science. . . . She doesn’t speak the language of science “like a native”; she is a native.

(Penelope Maddy)

> *As I see it, the goal of philosophy of mathematics is to interpret mathematics, and articulate its place in the overall intellectual enterprise. One desideratum is to have an interpretation that takes as much as possible of what mathematicians say about their subject as literally true, understood at or near face value. Call this the faithfulness constraint.* (Stewart Shapiro)

I agree with all of that, until the last word comes as a surprise.

Is it a *constraint* to accept the words of people whose ideas we interpret and articulate? Call it the *faithfulness opportunity*.

The best ideas about mathematics come from mathematicians!

Combinatorial mathematicians regard simple bijective proofs or partition identities as explanatory, in contrast with non-explanatory proofs by generating functions.

(Marc Lange)

Lange then quotes half a dozen combinatorists saying what he says they do.

I will come to other good examples. But it is not the norm.

Paul Benacerraf "What numbers could not be" begins with a quote:

> *The attention of the mathematician focuses primarily upon mathematical structure, and his intellectual delight arises (in part) from seeing that a given theory exhibits such and such a structure, from seeing how one structure is "modelled" in another, or in exhibiting some new structure and showing how it relates to previously studied ones.*
>
> (Richard Martin)

Benacerraf and Martin quote no examples, and name no possible examples but Hilbert and Tarski.

Hellman *Mathematics Without Numbers* and Shapiro *Philosophy of Mathematics: Structure and Ontology* discuss three examples: Dedekind, Hilbert, and Bourbaki.

With few actual quotes.

None of them is quoted much.

It is hard to know what kind of structuralism Hilbert is meant to support or illustrate.

The Bourbaki did give an extensive, explicit theory of structure.

They never used it. No one ever used it.

Indeed, only a sketch of it appeared in the first edition of the *Elements*.

The theory first appeared in 1958 in one of the last books the group published.

They invented it as an alternative to category theory which was the standard theory of structure in mathematics by that time.

Individual members of Bourbaki using category theory contributed far more to current structural mathematics in their own names than they did under the name of Bourbaki.

So I am starting a campaign against anonymous mathematics.

I want to urge that every time a philosopher says “mathematicians do X,” there should be a quoted, specific example of some mathematician doing X.

1. Mathematics is interesting. That is, actual, specific mathematics is interesting.
2. Specific, quoted examples can help clarify general philosophical claims.
3. Even a single example shows there is at least one example!
4. Readers can judge whether at least that example is being interpreted correctly.

Jessica Carter's work on structuralism becomes even more interesting and reliable when you know she is using her own research in topology.

She knows how the mathematicians thought of it, since she is one of the mathematicians!

Naturalism and philosophical modesty.

> *The naturalistic philosopher begins his reasoning within the inherited world theory as a going concern. He tentatively believes all of it, but believes also that some unidentified portions are wrong. He tries to improve, clarify, and understand the system from within.* (Willard Quine)

This sense of naturalism is fine for physical sciences.

Only a few people, like Luitzen Brouwer later in his career, believe major parts of standard mathematics are actually wrong.

Conclusions by Lie, Gordan, Poincaré were accepted as true. And were true (mostly).

Dedekind and Kronecker both believed mathematics was unacceptably sloppy and misunderstood.

I count Dedekind and Kronecker as naturalist philosophers in mathematics since both pursued the stated goals of Quine’s naturalistic philosopher.

They improved, clarified, and understood mathematics in philosophic ways – by looking to what lies centrally within the subject.

I follow Penelope Maddy.

A philosopher is someone who confronts “traditional metaphysical questions of what there is” and how we know it.

A *second philosopher* is naturalistic in the sense of being actively involved with some science and looking for the philosophical answers in existing scientific methodology:

> *All the Second Philosopher’s impulses are methodological, just the thing to generate good science. . . . She doesn’t speak the language of science “like a native”; she is a native.*

Dedekind and Kronecker fit this description. So do Poincaré, Hilbert, Weyl, Reuben Hersh, Mac Lane, and William Lawvere, for example.

Weyl is an especially compelling example.

At first was no naturalist at all but attempted an external, philosophically motivated reform of mathematics.

He gave up that philosophy on the naturalistic, methodological grounds it did not generate good mathematics.

Not every philosophically important mathematician is a philosopher.

Emmy Noether and Alexander Grothendieck are silent on traditional philosophic questions.

Brouwer shows that not every mathematician philosopher is naturalistic.

But mathematicians who address philosophy are commonly second philosophers and in that sense naturalistic.

Many philosophers including Maddy associate naturalism with philosophical modesty.

But I call Dedekind, Hilbert, and other mathematicians philosophers.

I do not think they should have been modest about the nature of mathematics!

I do not think they were wrong to deliberately alter the practice of mathematics.

Rather than requiring modesty, this kind of naturalism requires philosophers to be informed.

Dedekind and Kronecker both set out to reform the practice of mathematics along philosophic lines.

Both had lasting good effects.

Dedekind's philosophy won decisively in the long run, so it is tempting to say it was not philosophy.

We are tempted to say true ideas about math are just math, and only mistakes about math can be philosophy!

Frege and Russell set out to reform mathematics along philosophic lines, with very good effects, as did Brouwer (in topology), and Mac Lane.

Philosophy does not need to be modest, it just needs to be right!