New 20th century working methods

A. Geometrizing number theory:
Hilbert and Hurwitz to Weil

B. Why Bourbaki wrote the *Elements*

C. Unifying topology and group theory:
Hopf, Eilenberg, Mac Lane, and Steenrod

Geometrizing number theory

Hilbert and Hurwitz to Weil

Of course such a notion [of purity of method] did not enter my head. I was left with a totally different impression.

(André Weil)

André Weil, a mathematician's mathematician.

Raised as a genius, the same as his younger sister Simone Weil (famous pacifist religious philosopher).

Youngest person allowed to take the entrance exam for the École Normale Supérieur. (ENS).

The greatest mathematician of the middle 20th century.

Went on vacation in the mountains with his family, when he was 16.

Simone said this mountain landscape impressed the notion of purity upon her soul. No such notion entered my head. [Instead], seeing the shafts of sunlight crisscross at a distance at sundown gave me the idea of composing on several planes simultaneously. . . . It came to me there, in the mountains, whereas in my sister these same landscapes inspired radically different reflections.

I. André Weil

A. Utterly unified view of mathematics.

B. Disdains purity of method.

In Serre's words Weil 1928 dealt with:

Diophantine equations, that is to say, with rational points on algebraic varieties.

At that time, the only known method was Fermat's descent; very often, the application of this method depended on explicit calculations, so that a different little miracle seemed to happen in each particular case.

Weil was the first to see that behind these computations there was a general principle . . . a sort of transfer between algebra (in principle easy) and arithmetic (harder).

Weil adopts the geometric idea from Hilbert and Hurwitz (1890) and independently Poincaré (1901).

In such hands, geometry “gives simple means to completely resolve . . . a crowd (*foule*) of particular problems” (Weil 1928).

Weil remarks that this approach identifies new crucial features: *genus* of a curve and more.

Weil himself unifies the little arithmetic miracles.

Integer solutions to $A^2 + B^2 = C^2$:

$$3, 4, 5 \qquad 5, 12, 13 \qquad 7, 24, 25$$

Equivalent to rational solutions to $x^2 + y^2 = 1$:

$$\frac{3}{5}, \frac{4}{5} \qquad\qquad \frac{5}{13}, \frac{12}{13} \qquad\qquad \frac{7}{25}, \frac{24}{25}$$

Recall Kronecker's idea to eliminate fractions (and algebraic irrational numbers) by using more variables.

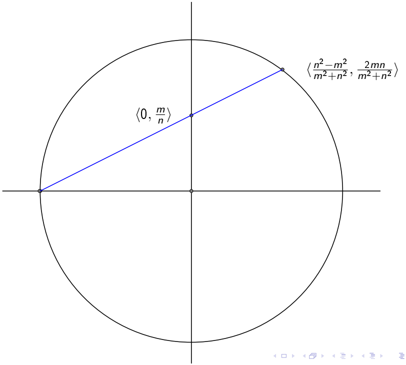
We are looking at rational points on the unit circle

$$S^1 = \{\langle x, y\rangle \mid x^2 + y^2 = 1\}.$$

There is a general formula:

$$x = \frac{n^2 - m^2}{m^2 + n^2} \qquad y = \frac{2mn}{m^2 + n^2}.$$

Geometrically, m/n is the slope of a line from $\langle -1, 0\rangle$. The point is the other intersection with the circle.


$\left\langle \frac{n^2-m^2}{m^2+n^2}, \frac{2mn}{m^2+n^2} \right\rangle$
$\left\langle 0, \frac{m}{n} \right\rangle$

People had long known this works for other conics.

Hilbert and Hurwitz saw in 1890 it is *not* about the degree of the equation, but the geometry of the curve.

It works as well for a singular cubic, say $y^2 = x^2 + x^2$.

Here too the rational points can be parameterized by rational slopes of lines (this time, through the singular point).

Poincaré noticed a similar method finds rational points on non-singular cubics, say

$$y^2 = x^3 + 3x^2 + x$$

Poincaré 1901 organized this beautifully, and showed that procedure works just as well for a singular degree 4 curve (or doubly singular degree 5. . .).

Poincaré conjectured a nice result for rational points on a torus – and on curves of higher genus (higher degree, minus singularity).

(For all this see Myles Reid's provocatively titled *Undergraduate Algebraic Geometry*.)

Weil would go farther then anyone before to unify geometry with number theory.

On his mountain vacation at 16

> *I would often stop to open a notebook. of calculations on diophantine equations. The mystery of Fermat's equation attracted me, but I already knew enough about it to realize that the only hope of progress lay in a fresh vantage point. At the same time, reading Riemann and Klein had convinced me that [cetain new geometry] had to be brought to the foreground. . . .*
>
> *My calculations showed me that Fermat's methods, as well as his successors', all rested on one virtually obvious remark*

to wit:

If $P(x, y)$ and $Q(x, y)$ are homogeneous polynomials algebraically prime to each other, with integer coefficients, and x, y are integers prime to each other, then the numerical values $P(x, y)$ and $Q(x, y)$ are almost prime to each other.

That is to say, their GCD admits a finite number of possible values.

Polynomials $x + y$ and $x - 2y$ are algebraically prime.

For all relatively prime m, n: $\mathrm{GCD}(m + n, m - 2n)$ divides 3.

So either the greatest common divisor of the values is 1, so $m + n$ and $m - 2n$ are relatively prime.

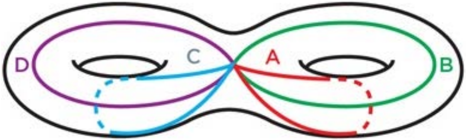
Or the greatest common divisor of the values is 3, so $(m + n)/3$ and $(m - 2n)/3$ are relatively prime.

Weil proved Poincaré's conjecture by a geometric proof, with a huge bonus.

Weil's proof is so simple (logically, not psychologically!) it works over any number field in place of the rational numbers!

The generality comes unsought.

Weil's proof uses complex analysis, specifically integration on Riemann surfaces.

$\Sigma_2 =$ 

Complex integration on the genus 2 Riemann surface Σ_2 is controlled by integration along the cycles A,B,C,D.

It becomes addition on the *Jacobian* $J(\Sigma_2)$, a 2 complex-dim torus, a quotient $\mathbb{C}^2/\Lambda$.

A genus n Riemann surface has Jacobian (analytically) an n complex-dim torus.

There was a problem.

Serre: “The algebraic geometry of the time had not yet developed the tools that were needed. Fortunately, Weil had read the works of Riemann at the École Normale, and he was able to replace the missing algebra by analysis: theta functions.”

Weil was utterly uninterested in “purity of method.” that is separating algebra/arithmetic from analysis.

Weil 1928 says of theta functions:

"For the properties of theta functions that I use, see for example, Krazer-Wirtinger, Enzykl. d. math. Wiss. II, B. 7."

Waves his hand at a 271 page encyclopedia article.

He neither knows nor cares to know which results he used.

To be clear: he knows all the results in the article. Does not care which are implicit in his thesis.

Lots of applications to Diophantine equations.

The whole subject was virtually incomprehensible among top mathematicians by 1928, when Weil proved it.

Perhaps the only people in the world who could read it were Carl Ludwig Siegel in Germany, and Louis Mordell in England.

And Mordell did not like Weil's style.

The faculty in Paris took Weil's word that his dissertation was correct.

Weil went much farther connecting geometric analysis to arithmetic.

Serre describes masterful cases.

In some, “Weil had to be content with partial results, partly unproved but which would turn out to be essentially correct; a fortiori, he could make no arithmetic application. . . .his work on Riemann-Roch served as a model to others fifteen years later.”

Where analysis did work, Weil would not *purify* it away.

When vast territories are being opened up, nothing could he more harmful to the progress of mathematics than a literal observance of strict standards of rigour. . . . At the same time, it should always be remembered that it is the duty, as it is the business, of the mathematician to prove theorems, and that this duty can never be disregarded for long without fatal effects.

Why Bourbaki wrote the *Elements*

There is no question of Bourbaki's containing anything original. Bourbaki does not attempt to innovate mathematics, and if a theorem is in Bourbaki, it was proved 2, 20, or 200 years ago. (Jean Dieudonné)

Imagine you are a mathematics student in France in 1922.

You have been admitted to the École Normale Supérieur, the best university in France – and thus the best in the world, according to you!

Your teachers are the best of their generation. But they are all over 60.

The generation of 35-55 year olds was lost to war.

Mathematics done after about 1900 is not taught.

Germany protected their professors (though not their students).

Leading classmates have travelled to Germany and studied there. Maybe you too.

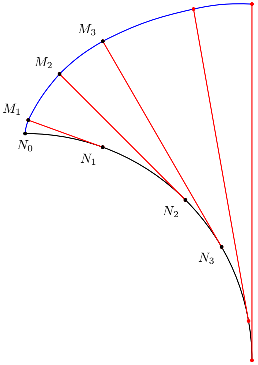
But France should lead the world in all things!

You learned analysis from a truly great textbook, Édouard Goursat's 1902 *Cours d'Analyse Mathḿatique*.

It taught interesting problems.

(Much more advanced than calculus books today.)

It had clear proofs –
except that it was vague about some concepts like *numbers, variable numbers, dependent variables* and *functions*.



If the velocity of some particle depends on time and position then $\partial v/\partial t$ and $\mathrm{d}v/\mathrm{d}t$ are different, because:

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial v}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial v}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}$$

Hel Braun: “In my student days university mathematics rested strongly on being mathematically gifted.”

Every member of Bourbaki was gifted!

But they saw that the more systematic, logical German approach was producing better mathematics.

Monday, 10 December 1934, Café Capoulade, Paris.

Weil invites 10 of his friends, all ENS graduates, to collaborate on a definitive new analysis textbook.

Not separate chapters by separate experts.

The whole group will write every part of it, in many drafts.

It will match the rigor of the Germans.

Notably van der Waerden's *Moderne Algebra*, based on lectures by Emmy Noether nd Emil Artin.

They quickly see this is impossible.

To reach the level of Goursat's *Cours d'Analyse*, at the new level of rigor, will require utterly reorganizing mathematics.

They must first do *set theory*, then *algebra*, then fundamentals of *real analysis*. . . .

The *Élements de Mathématique* became a 10 volume series, was never actually finished.

By 1960 mathematics had moved on – due in large part to the success of the *Élements de Mathématique*

Bourbaki:

1. Established a uniform terminology in algebra, analysis, and topology.
2. Put all the branches of mathematics into one logical order.
3. Established – not logical foundations – but uniform working foundations in set theory.

Most important: Bourbaki re-shaped mathematics education.

Students learn methods. Linear algebra, topology, Galois theory.

Few mathematics majors today could do the problems in Weber's 1895 *Algebra*, or Goursat's 1902 *Cours d'Analyse*.

But with a modern education, plus 15 minutes on line, you could learn to do any one of them.

Bourbaki also gave a *theory of structures* in 1958, which they never used.

It does not focus on concepts *useful* for proving theorems.

It was not general enough to include important structures actually in use by the 1950s.

Few people have ever read it.

Category theory was already around.

Samuel Eilenberg, who co-invented category theory with Saunders Mac Lane in the 1940s, was a member of Bourbaki in the 1950s.

Other members of Bourbaki were using it through the 1950s, and inventing new uses for it: Charles Ehresmann, Henri Cartan, Pierre Cartier, Alexander Grothendieck.

Why did Bourbaki not use category theory?

First, it was impossible to write a comprehensive *Elements of Mathematics* based on category theory, because it is impossible to write a comprehensive *Elements of Mathematics* at all.

It was a brilliant idea, and very good for mathematics. But it could not really be done.

Categories and functors in the 1950s were developed for immediate applications, and not as a general theory of structure.

The general idea of structure was exploding at that time, largely using categories and functors.

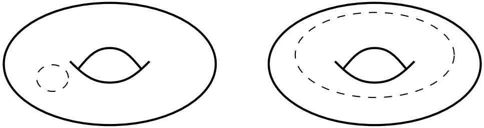
It was very much easier for Bourbaki to develop a theory that did not work, than to unify a sprawling mass of working methods into a theory.

C. Unifying topology and group theory
Hopf, Eilenberg, Mac Lane, and Steenrod

WHAT???? (WHO???)

The sweeping general theories of homology, homotopy, and category theory grew up to organize quite concrete information – by showing how to throw out most of it.

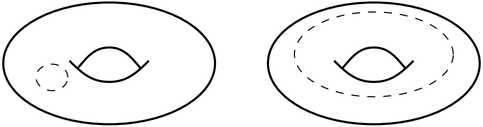
A small geometric distinction was important already to Riemann in 1850, and is still being explored today.



The cut on the left is somehow trivial, unrevealing. It does not show anything about the shape of the torus.

The on the on the right reveals the hole through the center of the torus.

But exactly what makes that cut *trivial*?

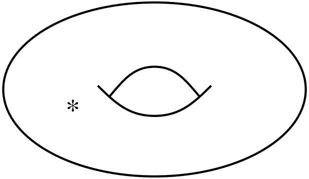


One kind of triviality: The cut on the left cuts the torus in two. The one on the right does not.

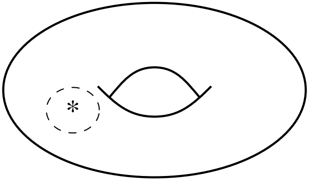
A different kind of triviality: The cut on the left is *contractible*. The one on the right is not.

Riemann 1850 already had to distinguish these two different ways of being “trivial.”

Consider a *punctured* torus, where the mark is a small hole



This cut on the punctured torus still cuts it in two:



But it is *not* contractible.

Every contractible curve cuts a surface in two. But not every curve that cuts it in two is contractible.

Poincaré saw these two senses of "triviality " give two ways to make the curves in a topological space into a group.

We will start with the easier one.

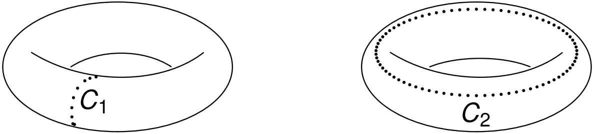
Poincaré defined addition for curves so that $C + C'$ is the union of C and C' while $-C$ corresponds to reversing the direction of C.

Poincaré said a closed curve C on the torus – or any surface S – is *homologous to zero*, and wrote $C \sim 0$, if it cut out a piece of the torus.

If a list of closed curves $C_1, C_2, \ldots, C_k$ formed the total boundary of some part of the surface then he would write

$$C_1 + C_2 + \cdots + C_k \sim 0$$

and say the sum is homologous to 0.



Then every curve on the torus is homologous to a sum $aC_1 + bC_2$ for some integers a, b.

The *first homology group* of the torus, $\mathrm{H}^1(T)$ is the group $\mathbb{Z} \times \mathbb{Z}$ of pairs of integers $\langle a, b \rangle$.

For any topological space S, the first homology group $\mathrm{H}_1(S)$ is the group of curves on S *modulo* the curves homologous to 0.

In other words it is the curves on S where we ignore any that are boundaries of surfaces in S.

Everyone who was interested in the subject also knew that for every continuous function $f : S \to S'$ between spaces, there is a group homomorphism $\mathrm{H}_1(f) : \mathrm{H}_1(S) \to \mathrm{H}_1(S')$ between the groups.

And everyone after Poincaré knew about higher dimensional analogues where $\mathrm{H}_n(S)$ is the group of n-dimensional submanifolds of S, modulo a similar homology relation.

But instead of ignoring all curves in S homologous to 0, we could ignore only those that are contractible.

This gives the *fundamental group* of S, called $\pi_1(S)$.

The group of all ways a curve tangle around in S.

Much more informative than $H_1(S)$. Much harder to calculate.

Again, for every continuous function $f : S \to S'$ there is a group homomorphism $\pi_1(f) : \pi_1(S) \to \pi_1(S')$.

Every group G is (isomorphic to) the fundamental group of some space(s) T, $G \approx \pi_1(T)$.

Vietoris, and Alexandroff, and Hopf showed that the best way to see how continuous functions $f : S \to S'$ between spaces produce group homomorphisms $\mathrm{H}_n(f) : \mathrm{H}_n(S) \to \mathrm{H}_n(S')$ is to use Noether's algebra.

This led to an explosion of many different way to define the homology groups $\mathrm{H}_n(S)$ plus a variant clalled *cohomology groups* $\mathrm{H}^n(S)$ which first served to clarify Poincaré duality.

New theorems and methods poured in faster than anyone could follow.

People would use each others results, while using different definitions!

Everyone felt this “should be okay” (most of the time).

Then two things happened that led no a new and better organization of the ideas.

Heinz Hopf realized it can be valuable to study a group G by finding a space with $\pi_1(S) \approx G$ and looking at the cohomology groups $\mathrm{H}^k(S)$.

And *much better*, you can find those cohomology groups directly from G without finding any space!

So you can call them cohomology groups $\mathrm{H}^k(G)$ of G. This *group cohomology* quickly found uses in number theory.

At the same time (separated by the war) Mac Lane was working with the same ideas as Hopf, using Noether's factor sets.

> *I had calculated an example of the group of group extensions for an interesting factor group involving a prime number p. When I told Sammy this result, he immediately saw that it answered a question of Steenrod about the regular cycles of the p-adic solenoid.*

Very specific problems led to a three very general theories: category theory, axiomatic (co-)homology of topological spaces, group cohomology.