Current philosophies

A. Structuralist philosophers of mathematics

B. Mac Lane the last mathematician from Hilbert's Göttingen

C. Lawvere's categorical foundations

Structuralist philosophers of mathematics

We without doubt possess creative power.

(Dedekind)

A lot of this comes from a joint article with Liu Jie that will appear in *The Journal of Dialectics of Nature*.

Three important structuralist philosophers of mathematics: Richard Dedekind, Geoffrey Hellman, and Stewart Shapiro.

I agree with Dedekind, but his ideas are not very far developed.

The following lectures argue the two most developed structuralist philosophies come from Saunders Mac Lane and William Lawvere.

I think Dedekind is right when he insists we create mathematical objects:

> *We are a godlike species and without doubt possess creative power not merely in material things (railroads, telegraphs), but quite specially in intellectual things.*

Like Kronecker, and Gordan, Dedekind was not serious about god here.

And like them, he was serious about mathematics.

I will argue that we create mathematical objects just as we create poems, and governments.

Poems are real. Governments are real. We really create them.

The complex numbers are real. We really created them.

That Dedekind quote is from his letter to Weber, who had suggested that we could *define* the natural numbers the way Frege and Russell later would, as certain classes of classes; and define real numbers as Dedekind cuts on the rational numbers.

Many analysis textbooks do define real numbers as Dedekind cuts today.

Dedekind disagreed:

> *I should still advise that by natural number... there be understood not the class... but rather something new (corresponding to this class), which the mind creates.... Where you say that the irrational number is nothing else than the cut itself, I prefer to create something new (different from the cut).*

The logician John Mayberry, who I admire very much though we disagreed very much, proposed a test for this approach.

He meant to refute Dedekind.

He compared Dedekind’s idea to the character Owen Glendower in Shakespeare’s play *Henry V part 1*.

Glendower claimed he was a magician. His son-in-law Hotspur disagreed.

Glendower:I can call spirits from the vasty deep [ocean].

Hotspur. Why, so can I, or so can any man. But will they come when you do call for them?

So I ask: When Dedekind created his real numbers, did they work?

I think Glendower was a fool. And Dedekind was right!

Dedekind knew there was a serious problem about logic.

But, like Poincaré he considered it a question for logicians, not a challenge to mathematics.

Hellman's and Shapiro's structuralism arise form philosophy rather than mathematics.

Paul Benacerraf's *What numbers could not be*' says (as Dedekind did before)we do not normally assign set theoretic properties to numbers.

He calls for a theory of *abstract structures* which differ from ZF sets in that

> *the "elements" of the structure have no properties other than those relating them to other "elements" of the same structure.*

Shapiro offers a structure theory meeting Benacerraf's demand.

It distinguishes between a "structure" and "the systems that exemplify it."

Abstract "structures are bona fide objects," which means they do exist. And, as Benacerraf asked, their elements have no properties except relations to each other in the structure.

Systems are more concrete: Shapiro says

> *the natural number structure exists independent of any concrete system that may exemplify it.*

I argue that Shapiro's distinction of systems and structures does not help to understand current practice.

Two proposals:

1. Structure=object in a category. System=object in a concrete category. So it is a difference in our attention, not in the objects.
2. Rather than coherence of second order logic, rely on the coherence of current mathematics itself.

Explain *concrete category*. Like coordinates on a physics problem or other space. It expresses objective facts, but the particular choice of coordinates is partly arbitrary.

Second item was not really possible in 19th century since published math at that time was *not* very coherent. But 150 years of articulating the relations between different schools of math, and different branches, and logical foundations, has made it possible.

The textbook treatment of complex numbers.

Conway and Smith define a complex number as an expression $x_0 + x_1 i$.

Is this concrete, or abstract?

Shapiro has told me it would depend.

1. If Conway has a concrete definition of "expression," then his complex numbers form a system.
2. If Conway has an abstract/structural definition of "expression," then his complex numbers form a structure.

Conway probably never thought about whether expressions are abstract or concrete.

Foundational issue: this only seems to arise if sets are like ZF sets.

Mathematical issue: Are you looking at just as algebraic (say, in the category of real algebras)?

That is are you looking at algebraic relations of $\mathbb{C}$ to $\mathbb{R}, \mathbb{C}[X, Y]$ etc?

Or do you also want to work with the *set* of complex numbers – so you want the concrete category $\mathbf{U} : \mathbf{RealAlg} \to \mathbf{Set}$.

It is not a question about $\mathbb{C}$. It is a question of what you want to do.

On categorical foundations everything is a structure.

Many structures also have underlying sets.

Most ways of defining a group G give a natural choice of set as $\mathbf{U}G$.

Even the more abstract ways give a functor $\mathbf{U}\colon \mathbf{Grp} \to \mathbf{Set}$ defined up to isomorphism (or equivalence).

We do not identify G with $\mathbf{U}G$ but we use both for different things.

A system can be a structure plus a chosen underlying set.

Shapiro's paper "Structure and identity" says Helman is not faithful to practice:

> *on the eliminative program, numerals are understood to be bound variables, perhaps under a modal operator. This does not accord with faithfulness.*

I do not believe numerals as bound variables is far from practice. Shapiro gives no examples, so I do not see why he thinks it is.

By this standard, Shapiro's system/structure divide is far from faithfulness.

It is impossible to apply to ordinary textbooks. A different distinction that is used in practice answers the questions better.

As to what structures are, Shapiro says

> *The main principle behind [my] structuralism is that any coherent theory characterizes a structure, or a class of structures. . . . If Φ is a coherent formula in a second-order language, then there is a structure that satisfies Φ. . . . it is far from clear what "coherent" comes to here.*
>
> *There is no getting around this situation. We cannot ground mathematics in any domain or theory that is more secure than mathematics itself. All attempts to do so have failed, and once again, foundationalism is dead*

Foundationalism in the sense of a theory more secure than mathematics was dead to Russell in 1919.

Foundationalism in the sense of proposals to organize mathematics will never die as long as there is math.

Further, Shapiro’s idea of second order logic is independent of set theory.

It has nothing close to the actual security of existing math.

Mac Lane and Lawvere will both ground math on math itself.

Geoffrey Hellman offers a variant structuralism where abstract structures are not things.

Rather structuralist mathematics focusses its attention of structural properties, *isomorphism invariant* properties.

The objects are ZF sets, but number theorists do not ask about individual sets that satisfy the Peano axioms. They ask what is common to all sets that satisfy the Peano axioms.

The abstract *structure of* the natural numbers is just what is common to all of those non-abstract sets.

Certainly mathematicians work with isomorphism invariant properties.

Hellman ties this to modal logic.

He wants to solve the problem of existence of mathematical objects by saying they do not *actually* exist.

They only *possibly* exist.

The theory makes formal logical sense.

But I do not think it solves any problems.

I know what it means to say “I could sing the song 小苹果 , but I won’t.”

I do not know what it means to say “The number π could exist, but it does not.”

Hellman relates his modal structuralism to the freedom of mathematics:

> *On the sort of view we should like to articulate, mathematics is the free exploration of structural possibilities, pursued by (more or less) rigorous deductive means.*

I believe this is an important misunderstanding of the freedom of mathematics.

For Dedekind’s friend Cantor the freedom of mathematics is not about what might be, but what must be:

> *Mathematics is entirely free in its development and bound only by the obvious consideration that its concepts must be noncontradictory in themselves, and must have fixed relations, ordered by definitions, to previously formed, already existing concepts. . . . Once a [new concept of] number satisfies all these conditions, it can and must be considered as existent and real in mathematics.*

He does not say infinite numbers are possible!

They “can and must be considered existent and real in mathematics.”

When János Bolyai asked if a non-Euclidean geometry was possible, he was not asking a question of modal logic.

And he would not have accepted an answer saying: “Yes, it is possible, but not actual.”

He wanted actual axioms, and concluded the geometry was real.

He told his father: “I created a new, different world out of nothing.”

More or less the same is true of Stanisław Ulam when he asked if there could be a measurable cardinal.

Hellman created some confusion about categorical structuralism.

He naturally says categorical foundations should answer the question "*What categories or toposes exist?*"

He repeats more formally: "*what axioms govern the existence of categories or toposes*?"

He claims "this problem as it confronts category theory can be put very simply: the question really just does not seem to be addressed!"

Linnebo and Pettigrew repeated this claim in a 2010 article.

It would be a serious problem, if it were true.

Actually Bill Lawvere gave the first formal, axiomatic answer in the *Proceedings of the National Academy of Sciences of the USA* in 1965.

Hellman relies on Solomon Feferman's 1977 paper "Categorical foundations and foundations of category theory."

Feferman refers to several published answers to Hellman's question.

But Feferman says details are "irrelevant to my argument," because "the objections here to a program for the categorical foundations of mathematics apply" to any axioms using categories.

Feferman said you do not need to read any of the published category theoretic axioms, and somehow Hellman, Linnebo, and Pettigrew thought none had been published.

The next lecture comes to the first published categorical foundation for mathematics.

Saunders Mac Lane
the last mathematician from Hilbert's Göttingen

We must know. We will know. (David Hilbert)

Saunders Mac Lane grew up knowing, and believing the new 20th century mathematics of logic and set theory.

Wanted to read *Principia Mathematica* around 1928 at Yale.

His teacher instead assigned Felix Hausdorff's 1914 book *Set Theory*.

So he learned semi-formalized set theory as a working foundation.

Got the highest grade point average at Yale.

Heard that Emmy Noether in Germany was good for algebra.

Went to graduate school at Chicago (without applying!)

Completed his doctorate, in logic, at Göttingen, 1934 (should have been with Paul Bernays, but Bernays was Jewish, so finally it was with Hermann Weyl).

He and Bernays attended lectures by Noether together.

Noether invented *factor sets* to replace huge arithmetic calculations by conceptual:

> *I personally did not understand factor sets well at the time of Noether's lectures, but later Eilenberg and I used factor sets to invent the cohomology of groups.*

Calculation is always the basis of number theory but when these idea reduce the calculations for any given problem, even bigger problems became feasible.

He heard public lectures by Hilbert, saying “we must know, we will know.” And learned Hilbert’s idea of metamathematics.

He studied philosophy with Moritz Geiger and talked about it with Hermann Weyl.

He learned a lot from Weyl. But Weyl admired Brouwer’s intuitionism. Mac Lane found Brouwer “pontifical and often obscure.”

His dissertation aimed at

> *structure theory for Mathematics based on the principle of leading ideas*

He wanted to reveal the leading idea of each proof, and help to find new proofs, by abbreviating the logic of *Principia Mathematica* to shorten the proofs:

> *one can construct broader and deeper methods of abbreviation based on the concept of a plan of a proof. . . which efficiently determines the individual steps of the proof.*

The plan is not the leading idea, but *lets us see* the leading idea.

Quickly decided this would not work.

He lost interest in reforming mathematics by philosophic *theories*.

Kept wanting to reform mathematics, by math itself, and logic, and philosophic *insights*.

Always believed (good) proofs have *leading ideas* at their heart. (Geiger's influence?)

Wrote papers on logic, topology, graph theory, lots of new mathematical ideas.

Specialized in difficult algebra, related to Noether's.

Collaborated with Eilenberg, and created category theory.

Functors and natural transformations would not *be* the leading ideas of proofs, but would *help us see* leading ideas.

From the time he wanted to read *Principia Mathematica* as an undergraduate, to the end of his life, he never lost faith that the right fundamental concepts will give the best working methods.

A founding member of the Association for Symbolic Logic. He encouraged Stephen Kleene to write *Introduction to Metamathematics* and critiqued drafts. His doctoral students include logicians William Howard, Michael Morley, Anil Nerode, Robert Solovay and Steven Awodey.

In the 1950s Mac Lane was the head of the Mathematics Department at Chicago, vice president of the National Academy of Sciences and the American Philosophical Society, and president of the American Mathematical Society, and of the Mathematical Association of America.

He received the National Medal of Science in 1989.

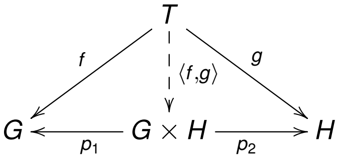
Up to 1950 categories were used to organize huge systems of groups and spaces.

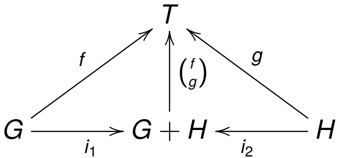
You did not “need” categories and functors for simple things like defining the product of two groups.

Mac Lane realized you could *use* categories and functors to clarify the relations between the *direct product* $G \times H$ and the *free product* $G + H$.

Algebraically $G \times H$ and the *free product* $G + H$ are quite different. Experts knew how to use each. But did not have good explanations of the difference.

Mac Lane explained that the definitions just reverse the arrows!





Mac Lane made this useful in relating homology to cohomology.

Categories and functors are not just for huge calculations.

They are also for explanations of small structures.

Grothendieck extended these ideas as the basis of his 1957 Tōhoku approach to cohomology.

In 1963 he met graduate student Bill Lawvere, who had an idea that you could describe sets using category theory.

Saunders told him to forget it.

Bill give him a paper, Saunders read it on an airplane, and then got it published in the *Proc. of the National Academy of Sciences*.

This revived Sanders's earlier interest in logic and philosophy.

In philosophy, Mac Lane kept an idea he learned in Göttingen:

> *the real world is understood in terms of many different Mathematical forms*

These forms exist in mathematics, but

> *Mathematical existence is not real existence*

because he believes only falsifiable, empirical science can be true.

Mathematics is not empirically falsifiable, so it is not true. But it is "correct."

Mac Lane views foundations as “proposals for the organization of Mathematics.”

He believes we know mathematics before any foundation, and mathematics is based on sets.

We can axiomatize sets using ZFC, but he prefers Lawvere’s *Elementary Theory of the Category of Sets* (ETCS).

axiomatizing not elements of sets but functions between sets.

Mac Lane liked ETCS precisely because it is *not* a new conception of sets!

It is a new formalization of just what mathematicians all know about sets.

ETCS is "structural" in Benacerraf's sense, one year before Benacerraf's paper.

Elements of a set in ETCS have literally no properties except that they are elements of that set, each is itself, and none is any other.

Let $\mathcal{P}(x)$ be any formula in categorical set theory, with no constants and no occurrence of variables x or y.

Then the following statement is provable in ETCS:

$$(x \in S \mathbin{\&} y \in S) \;\Rightarrow\; [\mathcal{P}(x) \Leftrightarrow \mathcal{P}(y)].$$

Also, in ETCS isomorphic sets $S \cong S'$ provably have all the same properties.

Let $\mathcal{P}(X)$ be any formula in categorical set theory, with no constants and no occurrence of variables S or S'.

Let $\mathrm{Isom}(S, S')$ be the formula saying S and S' are isomorphic.

Then the following statement is provable in ETCS:

$$\mathrm{Isom}(S, S') \;\Rightarrow\; [\mathcal{P}(S) \Leftrightarrow \mathcal{P}(S')].$$

The ETCS axioms begin with the first order Eilenberg-Mac Lane category axioms, but instead of “object” and “arrow” we say “set” and “function.”

We can start with the axiom that there is a *terminal set* 1, that is set such that

$$\forall S\ \exists!\ f : S \to 1.$$

There may be many terminal sets, but all are isomorphic, and we call one of them 1.

Define an *element* $x \in S$ to be a function $x : 1 \to S$.

Immediately, 1 has exactly one element, namely $1_1 : 1 \to 1$.

So 1 is a *singleton set*.

Of course there may be many singleton sets.

And all are isomorphic.

The extensionality axiom says every function is fully determined by its effect on elements.

$$\forall\, f \neq g : A \to B,\ \exists x : 1 \to A\ [\, f(x) \neq g(x)\,].$$

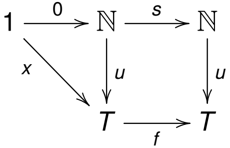
$$1 \xrightarrow{\quad x \quad} A \underset{g}{\overset{f}{\rightrightarrows}} B$$

Or, if you prefer, with free variables f, g:

$$(\forall\, x \in A,\ fx = gx) \to f = g.$$

The axiom of infinity: There is a set $\mathbb{N}$ with an element $0 \in \mathbb{N}$ and a function $s : \mathbb{N} \to \mathbb{N}$ such that:

$$\forall T \; \forall x \in T \; \forall f : T \to T \; \exists! u \; (u(0) = x \;\&\; us = fu)$$

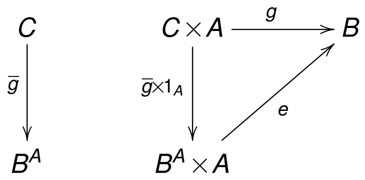


So $\mathbb{N}$ has elements $0, \; 1 = s(0), \; 2 = ss(0), \; 3 = sss(0), \; \ldots.$

Every two sets A, B have a product, as we have already defined.

There is a function set from each set A to each set B:

$$\forall\ C\ \forall g : C \times A \to B,\ \exists!\ \overline{g} : C \to B^A$$



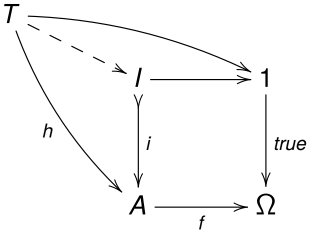
The axiom of (bounded) comprehension is the most unfamiliar to (non-categorical) logicians though the devices are familiar in geometry and measure theory.

It says there is a *truth value object* Ω with two distinct elements $\mathit{true}, \mathit{false} : 1 \to \Omega$ which can define subsets by *characteristic functions*.

1. Every function $\chi : A \to \Omega$ determines a subset $i : I \rightarrowtail A$ with $\chi(i) = \mathit{true}$.
2. Every subset $i : I \rightarrowtail A$ determines a function $\chi : A \to \Omega$ with $\chi(i) = \mathit{true}$.

The square here is a *pullback* if it has this property:

$$\forall h\colon T \to A\ [(fh = true) \to \exists! u(iu = h)]$$



The comprehension axioms says every one-to-one $i : I \rightarrowtail A$ determines a unique χ doing this; and every χ determines $I : I \rightarrowtail A$ uniquely it up to equivalence of subsets.

The axiom of choice says every onto function has a right inverse.

And that is all the ETCS axioms.

1. Eilenberg-Mac Lane axioms (but we say "sets and functions" inst4d of "objects and arrows."
2. There exists 1, and products.
3. There exist function sets.
4. The (bounded) comprehension axiom.
5. The axiom of choice.

It is a finitely axiomatized theory. Its consistency strength is well known (BZ, STT).

Lawvere's categorical foundations

It is not just that [Lawvere and Tierney] proved these things,
it's that they dared believe them provable. (Peter Freyd)

To understand discovery and invention in mathematics, and especially to understand categorical foundations.

You have to know that the easiest way to introduce a new idea is *usually not* the way it was discovered.

ETCS was not the first form of categorical foundations Lawvere conceived.

It is not his favorite (or mine). Though it is Mac Lane's.

Lawvere began with the *Category of Categories as Foundations* (CCAF).

He prefers this. And so do I (as i have said in print since 1991).

He got the idea of defining *function set* B^A categorically, described above, only as a special case of defining *functor categories* this way.

The first step came from methods of geometry. It was to see that objects, arrows, and composition in a category $\mathbf{C}$ correspond to functors

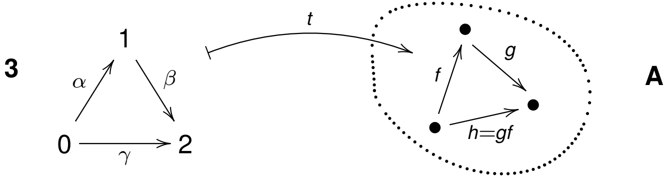
$$\mathbf{1} \xrightarrow{A} \mathbf{A} \qquad \mathbf{2} \xrightarrow{f} \mathbf{A} \qquad \mathbf{3} \xrightarrow{t} \mathbf{A}$$

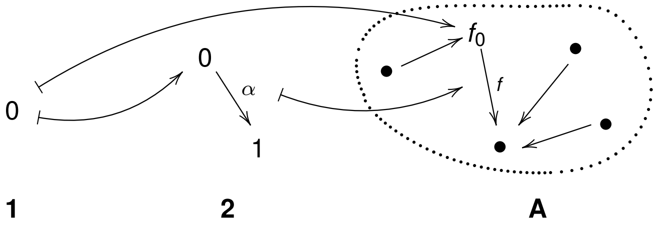
Compare the way that point of a space M are functions $p : 1 \to M$, while curves are functions $c : \mathbb{R} \to M$, and surfaces are functions $s : \mathbb{R} \times \mathbb{R} \to M$.

Okay points are only defined that way in some parts of algebraic geometry.

But curves and surfaces are often defined that way since 1800 and before.

Picture the mathematical world as a world of processes.

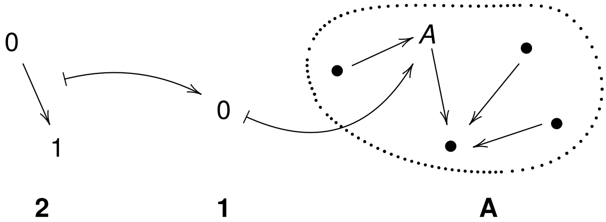




Given an object $A : \mathbf{1} \to \mathbf{A}$ its identity arrow in $\mathbf{A}$ is the composite

$$\mathbf{2} \longrightarrow \mathbf{1} \xrightarrow{A} \mathbf{A}$$

Or in a picture the single non-identity arrow of $\mathbf{2}$ is collapsed onto the single object of $\mathbf{1}$ and then to object A in $\mathbf{A}$:



Some philosophers and logicians are confused when Lawvere, or Mac Lane, or I say there are many different foundations that could be used as logical foundations for mathematics.

Feferman sad Mac Lane was “vacillating.”

Actually every logician has known it is true since long ago: Principia Mathematic, Zermelo-Fraenkel, Gödel-Bernays, Simple Type Theory, New Foundations, Aczel’s Anti-foundation axiom. . . .

But some people feel that when you talk about categorical foundations you should pick a side and stick to it!

I have said which I prefer.

It is one of the versions discussed on Lawvere’s 1966 paper on CCAF.

It is basic CCAF plus an axiom saying that one of the categories is **Set**.

Formally: begin with the Eilenberg-Mac Lane axioms but we say "categories and functors" instead of "objects and arrows."

There is a terminal category (or, singleton category) $\mathbf{1}$.

there is a category $\mathbf{2}$ that has exactly two functors $0 \neq 1 : \mathbf{1} \to \mathbf{2}$ and three functors $\mathbf{2} \to \mathbf{2}$.

Define: *object* A of $\mathbf{A}$ means $A : \mathbf{1} \to A$; and *arrow* f of $\mathbf{A}$ means

$f : \mathbf{2} \to A$.

The functor extensionality axiom says every functor is fully determined by its effect on arrows.

$$\forall\, \mathbf{F} \neq \mathbf{G} : \mathbf{A} \to \mathbf{B},\ \exists f : \mathbf{2} \to \mathbf{A}\ [\, \mathbf{F}f \neq \mathbf{G}f \,].$$

$$\mathbf{2} \xrightarrow{\ f\ } \mathbf{A} \underset{\mathbf{G}}{\overset{\mathbf{F}}{\rightrightarrows}} \mathbf{B}$$

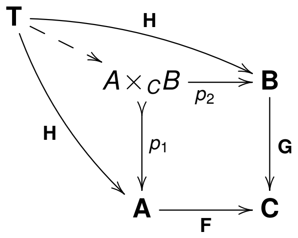
Or, if you prefer, with free variables $\mathbf{F}, \mathbf{G}$:

$$\big(\forall\, f : \mathbf{2} \to \mathbf{A},\ \mathbf{F}f = \mathbf{G}f\big) \to \mathbf{F} = \mathbf{G}.$$

Define composition of arrows f, g of $\mathbf{A}$ by using triangles $t : \mathbf{3} \to \mathbf{A}$.

Assert that every two functors $\mathbf{F} : \mathbf{A} \to \mathbf{C}$, $\mathbf{G} : \mathbf{B} \to \mathbf{C}$ have a pullback

$$\forall \mathbf{H} : \mathbf{T} \to \mathbf{A}, \mathbf{K} : \mathbf{T} \to \mathbf{B}\ [(\mathbf{FH} = \mathbf{GH}) \to \exists ! \mathbf{u}(p_1\mathbf{u} = \mathbf{H}\ \&\ p_2\mathbf{u} = \mathbf{K})]$$



Assert existence of functor categories.

So far these axioms do not prove much.

Now assert there is a category **Set** whose objects and arrows (as defined using **1**, **2**) sastify the ETCS axioms.

This implies a great many other large categories exist: **Grp**, **Ring**, $\mathbb{R}$**Alg** . . . with the usual functors among them.

All the mathematics of say Lang's *Algebra* and Munkres *Topology*. Enough Grothendieck toposes for all of cohomological number theory and algebraic geometry.

Project: These axioms give all commonly used *quotient categories*, such as *categories of fractions*, for categories that are closely linked to **Set**.

But they do not give quotients of categories unrelated to **Set**.

Is there a desirable extension of the axioms to do that?

I don’t know.

My categorical structuralism: “structures” are categories and functors in this theory of CCAF.