Composing models

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ABSTRACT. *We propose and study a new composition operation on (epistemic) multi-agent models with different vocabularies of propositional letters. This operation allows us to compose large models by small components representing agents' partial observational information. Our investigation provides ways to decompose (locally generated) epistemic models such that the truth of certain formulas are preserved. By using the composition operation we also propose and study action model composition and action model updates on models with arbitrary vocabularies.*

KEYWORDS: *dynamic epistemic logic, multi-agent models, model composition, agent observational power, action model update, epistemic model checking, interpreted systems, reduction techniques.*

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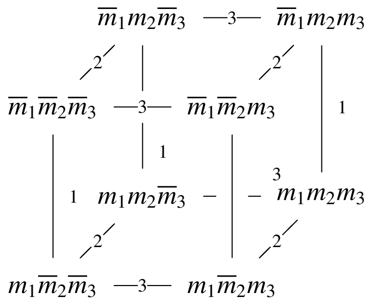
1. Introduction

The classic epistemic puzzle of Muddy Children reads as follows:

> Out of n children, $k \geq 1$ got mud on their foreheads while playing. They can see whether other kids are dirty, but there is no mirror for them to discover whether they are dirty themselves. Then father walks in and says: "At least one of you is dirty!". Then he requests: "If you know you are dirty, step forward now.". If nobody steps forward, he repeats his request: "If you now know you are dirty, step forward now.". After exactly k requests to step forward, the k dirty children suddenly do so.

The changes in the children's knowledge in this classic scenario can be modelled by the update mechanism of public announcements on the following initial Kripke model (when $n = 3$):



where m_i denotes that child i is dirty while $\overline{m}_i$ denotes that i is clean. The initial model for a muddy children scenario with n children has 2^n nodes to represent all the states of children's muddiness. As remarked in (van Benthem, 2009): *'There is no algorithm for producing it, but most people would agree that it fits the situation.'* We assume that people, even non-logicians, would be able to '*read off*' the information from a graph representation such as the representation given above. For it is clear from the picture that each agent does not know whether he is dirty or not, but is sure about the other two. In other words, once we have a model, anyone can check that it fits the requirements, but building such models is an *art*. Or maybe better: building such models is a combination of art and science. The difficult part is to turn a vague specification into a precise model that does justice to what the specification intends to say (but perhaps does not state explicitly).

In this paper we demonstrate how a bit more of the art of model construction can be turned into science by skillful use of appropriate building blocks. In particular, we show how such a model of muddy children can be viewed as a composition of n two-node models, each talking only about the muddiness of a single child. The intuition behind building models from components is that each component contains partial observational information about the whole model. For this, we introduce restricted multi-agent models, i.e., models with a limited set of 'relevant' propositional letters, we define a composition operation on restricted models and show that it is nicely behaved. Based on this notion of composition, we provide ways to decompose (locally generated) epistemic models such that the truth of certain formulas are preserved, thus suggesting methods to check relevant epistemic properties on small components of large models. By using the composition operation we also propose and study action model composition and action model updates on models with arbitrary vocabularies.

Dynamic epistemic logic (Gerbrandy, 1999; Baltag *et al.*, 1999; van Benthem *et al.*, 2006; van Ditmarsch *et al.*, 2007) has a somewhat monolithic architecture, and the decomposition perspective that we propose in this paper brings the framework closer to that of interpreted systems (Fagin *et al.*, 1995) (see (van Benthem *et al.*, 2009) and (Kooi *et al.*, 2010) for discussion). The intuition guiding interpreted systems is that each agent is following its own computational procedure, and that the internal states of the agents at different moments in time are given by local states. We claim that model components with restricted vocabularies can play a similar role in dynamic epistemic logic.

Related Work: Our composition of action models may be viewed as a notion of parallel composition of actions. The first concurrent operation in the framework of dynamic epistemic logic has been introduced in (van Ditmarsch *et al.*, 2003a; van Ditmarsch *et al.*, 2003b), where the authors follow the treatment of concurrency as in concurrent PDL (Peleg, 1987). The concurrent operator $\cap$ as in (van Ditmarsch *et al.*, 2003a) essentially splits the system into copies with each copy executing a concurrent component (see also (van Ditmarsch *et al.*, 2007, Chapter 5) for details). In some sense, composing actions in concurrent DEL may be viewed as merging agents who are acting in different ways, while in this paper we focus on merging propositional information which is distributed among agents in both static and action models. Compared to the large body of research about parallel compositions in various process algebra frameworks (e.g., (Milner, 1982; Bergstra *et al.*, 1985; Brookes *et al.*, 1984; Groote *et al.*, 1994)), the distinct feature of our operator is the merging of different vocabularies and preconditions. The restriction to epistemic models (S5 models) also gives specific results that are meaningful in the epistemic setting.

2. Composing epistemic models

Let P be a finite set of propositional letters. A multi-agent epistemic model $\mathcal{M}$ for P is a quadruple (W, I, R, V) with W a non-empty set of worlds, I a finite set of agents, R a function that assigns to each $i \in I$ an equivalence relation R_i on W, and V a function that assigns to each $w \in W$ a subset of P. In the sequel, we use $W^{\mathcal{M}}, I^{\mathcal{M}}, R^{\mathcal{M}}, V^{\mathcal{M}}$ to denote the corresponding elements in the definition of $\mathcal{M}$. We sometimes write $w \xrightarrow{i}_{\mathcal{M}} v$ to denote $wR_i^{\mathcal{M}}v$ in a model $\mathcal{M}$.

A *vocabulary* is a subset Q of P. A model over a vocabulary Q is a multi-agent model (W, I, R, V) where V is a valuation satisfying $V(w) \subseteq Q$ for each $w \in W$.

A *restricted* multi-agent epistemic model is a quintuple (W, I, R, V, Q) such that (W, I, R, V) is a model over vocabulary Q.

If $\mathcal{M}$ and $\mathcal{N}$ are restricted multi-agent models *with the same agent set I and the same vocabulary Q*, then a relation Z between $W^{\mathcal{M}}$ and $W^{\mathcal{N}}$ is called a *bisimulation* if whenever wZv the following hold:

Invariance $p \in V^{\mathcal{M}}(w)$ iff $p \in V^{\mathcal{N}}(v)$,

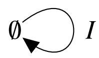
Zig if for some $i \in I$ there is a $w' \in W^{\mathcal{M}}$ with $w \xrightarrow{i}_{\mathcal{M}} w'$ then there is a $v' \in W^{\mathcal{N}}$ with $v \xrightarrow{i}_{\mathcal{N}} v'$ and $w'Zv'$,

Zag if for some $i \in I$ there is a $v' \in W^{\mathcal{N}}$ with $v \xrightarrow{i}_{\mathcal{N}} v'$ then there is a $w' \in W^{\mathcal{M}}$ with $w \xrightarrow{i}_{\mathcal{M}} w'$ and $w'Zv'$.

We say that $\mathcal{M}, w$ is *bisimilar* to $\mathcal{N}, t$ ($\mathcal{M}, w \leftrightarrows \mathcal{N}, t$) if there is a bisimulation linking w and v. We write $\mathcal{M} \leftrightarrows \mathcal{N}$ to indicate the existence of a *total bisimulation* (a bisimulation relation linking each state in $\mathcal{M}$ to some state in $\mathcal{N}$). A *bisimulation contraction* of a model is the model obtained by lumping together all its bisimilar states. A *Q'-restricted bisimulation* ($\leftrightarrows_{Q'}$)

between two models is a relation satisfies Zig, Zag and the Q'-restricted invariance condition to the restricted one: for all $p \in Q' : p \in V^{\mathcal{M}}(w)$ iff $p \in V^{\mathcal{N}}(v)$.

The unit model $\mathcal{E}$ for I is the restricted model $(\{e\}, I, \{\{(e, e)\} \mid i \in I\}, e \mapsto \emptyset, \emptyset)$. In a picture:



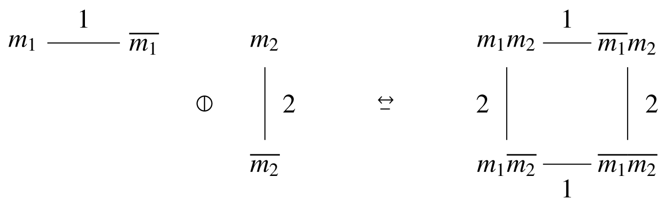
Definition 1. — *The* composition $\mathcal{M} \oplus \mathcal{N}$ *of two restricted multi-agent epistemic models* $\mathcal{M}, \mathcal{N}$ *with the same agent set* I *is given by* $(W, I, R, V, Q^{\mathcal{M}} \cup Q^{\mathcal{N}})$, *where*

– $W = \{(w, v) \mid w \in W^{\mathcal{M}}, v \in W^{\mathcal{N}}, V^{\mathcal{M}}(w) \cap Q^{\mathcal{N}} = V^{\mathcal{N}}(v) \cap Q^{\mathcal{M}}\}$
– $(w, v)R_i(w', v')$ *iff* $wR_i^{\mathcal{M}}w'$ *and* $vR_i^{\mathcal{N}}v'$
– $V((w, v)) = V^{\mathcal{M}}(w) \cup V^{\mathcal{N}}(v)$.

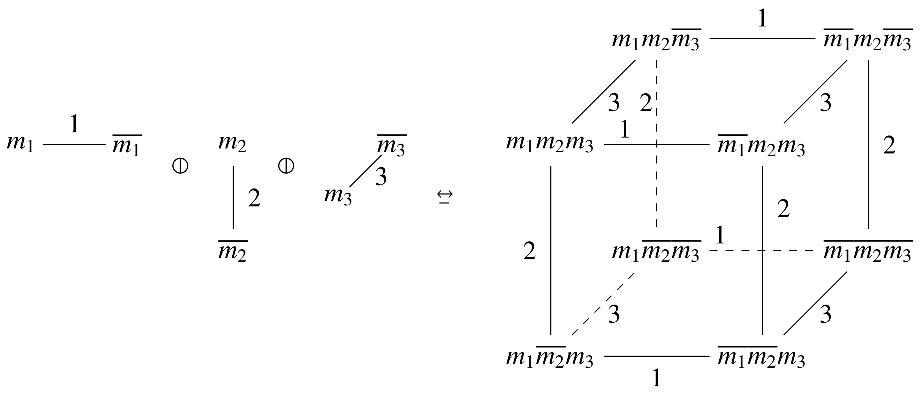
The new accessibility relations are defined as the product of the relations on the components, in the usual way, restricted to the pairs of worlds where the old valuations agree on the respective restricted vocabularies. Clearly the composition of two S5 models is still S5. Note that $V((w, v))$ agrees with $V^{\mathcal{M}}(w)$ on $Q^{\mathcal{M}}$ and with $V^{\mathcal{N}}(v)$ on $Q^{\mathcal{N}}$.

Remark 2. — The definition demands agreement on the values of basic propositions, and hence on purely Boolean formulas, or modal formulas of depth 0. This definition could be strengthened in obvious ways, by demanding agreement on modal formulas of depth 1, depth 2, and so on. This would boil down to considering 1-bisimilar pairs of worlds, 2-bisimilar pairs of worlds, and so on. In the current paper, we only study the case of 'Boolean' composition, but we believe that our definition extends smoothly to these more restricted cases. □

As a first example, here is a 'compositional version' of the muddy children scenario. Consider the following models, where each pair $m_i \stackrel{i}{\longleftrightarrow} \overline{m_i}$ represents a child that does not know whether it is muddy. We assume that each model $m_i \stackrel{i}{\longleftrightarrow} \overline{m_i}$ is restricted to $\{m_i\}$, and we leave out reflexive arrows (present for all agents).

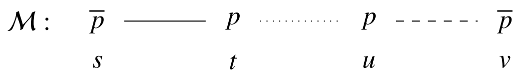


Intuitively, each 2-world model represents the children's observational power on whether child i is muddy, e.g., $m_i \stackrel{i}{\longleftrightarrow} \overline{m_i}$ captures the situation that child i does not know whether she herself is muddy while all the others do know whether child i is muddy. Composing with a third model gives:



And so on, for composition of multidimensional hypercubes with more and more children present.

Note that it does not generally hold that $\mathcal{M} \oplus \mathcal{M} \leftrightarrows \mathcal{M}$. In other words, the $\oplus$ operation is not idempotent. To see this, consider the following model:



where dashed lines, dotted lines and solid lines represent epistemic relations for three different agents respectively. It is clear that (t, u) is in the composed model $\mathcal{M} \oplus \mathcal{M}$, but according to the definition of $\oplus$, (t, u) is not related to a non-p world in the composed model. Therefore, (t, u) is not bisimilar to any world in $\mathcal{M}$. Still, Kripke models with restricted vocabularies form a commutative monoid:

Theorem 3. — *Restricted epistemic models over the same set of agents form a commutative monoid under the $\oplus$ operation, with total bisimilarity as the appropriate equality notion. In particular, we have:*

$$\begin{aligned} \mathcal{E} \oplus \mathcal{M} &\leftrightarrows \mathcal{M} \\ \mathcal{M} \oplus \mathcal{E} &\leftrightarrows \mathcal{M} \\ \mathcal{M} \oplus (\mathcal{N} \oplus \mathcal{K}) &\leftrightarrows (\mathcal{M} \oplus \mathcal{N}) \oplus \mathcal{K} \\ \mathcal{M} \oplus \mathcal{N} &\leftrightarrows \mathcal{N} \oplus \mathcal{M} \end{aligned}$$

Proof. — Commutativity and axioms about the unit model are immediate. We only check associativity here. Let $A(l)^y_x$ be the abbreviation of $V^x(l) \cap Q^y$ for $l \in \{w, v, k\}$ and $x, y \in \{\mathcal{M}, \mathcal{N}, \mathcal{K}\}$ where $w \in W^{\mathcal{M}}$, $v \in W^{\mathcal{N}}$ and $k \in W^{\mathcal{K}}$, e.g., $A(w)^{\mathcal{N}}_{\mathcal{M}}$ represents $V^{\mathcal{M}}(w) \cap Q^{\mathcal{N}}$. Thus the condition

$$\text{PC} := (A(w)^{\mathcal{N}}_{\mathcal{M}} = A(v)^{\mathcal{M}}_{\mathcal{N}} \text{ and } A(w)^{\mathcal{K}}_{\mathcal{M}} = A(k)^{\mathcal{M}}_{\mathcal{K}} \text{ and } A(v)^{\mathcal{K}}_{\mathcal{N}} = A(k)^{\mathcal{N}}_{\mathcal{K}})$$

expresses that w, v, k are pairwise compatible. A moment of reflection should assure that:

$$(w,(v,k)) \in W^{\mathcal{M}\oplus(\mathcal{N}\oplus\mathcal{K})} \iff \mathsf{PC} \iff ((w,v),k) \in W^{(\mathcal{M}\oplus\mathcal{N})\oplus\mathcal{K}}$$

Then it is straightforward to see that $\mathcal{M} \oplus (\mathcal{N} \oplus \mathcal{K}) \leftrightarrows (\mathcal{M} \oplus \mathcal{N}) \oplus \mathcal{K}$. ■

Note that $\leftrightarrows$ is indeed a congruence of this monoid:

Proposition 4. — *If $\mathcal{M}_1 \leftrightarrows \mathcal{M}_2$ and $\mathcal{N}_1 \leftrightarrows \mathcal{N}_2$ then $\mathcal{M}_1 \oplus \mathcal{N}_1 \leftrightarrows \mathcal{M}_2 \oplus \mathcal{N}_2$*

Proof. — Let Z_1 Z_2 be the total bisimulations witnessing $\mathcal{M}_1 \leftrightarrows \mathcal{M}_2$ and $\mathcal{N}_1 \leftrightarrows \mathcal{N}_2$ respectively. Then the relation $Z \subseteq W^{\mathcal{M}_1\oplus\mathcal{N}_1} \times W^{\mathcal{M}_2\oplus\mathcal{N}_2}$ defined by:

$$(w_1, v_1)Z(w_2, v_2) \iff w_1 Z_1 w_2 \text{ and } v_1 Z_2 v_2$$

is clearly a total bisimulation between $\mathcal{M}_1 \oplus \mathcal{N}_1$ and $\mathcal{M}_2 \oplus \mathcal{N}_2$. ■

This yields an algebraic preordering $\leq$ on the set of restricted multi-agent models:

$$\mathcal{M} \leq \mathcal{N} \text{ iff there is a } \mathcal{K} \text{ with } \mathcal{M} \oplus \mathcal{K} \leftrightarrows \mathcal{N}.$$

We proceed to give a structural characterization of this relation. For this, let a left-simulation between two restricted models $\mathcal{M}$ and $\mathcal{N}$ be a bisimulation with the invariance condition restricted to propositional letters in the vocabulary of $\mathcal{M}$, and without the zig condition (see the section on simulation and safety in (Blackburn *et al.*, 2001)). Formally, a left-simulation between $\mathcal{M}$ and $\mathcal{N}$ is a relation $C \subseteq W^{\mathcal{M}} \times W^{\mathcal{N}}$ such that wCv implies that the following hold:

Restricted invariance $p \in V^{\mathcal{M}}(w)$ iff $p \in V^{\mathcal{N}}(v)$ for all $p \in Q^{\mathcal{M}}$,

Zag If for some $i \in I$ there is a $v' \in W^{\mathcal{N}}$ with $v \xrightarrow{i}_{\mathcal{N}} v'$ then there is a $w' \in W^{\mathcal{M}}$ with $w \xrightarrow{i}_{\mathcal{M}} w'$ and $w'Cv'$.

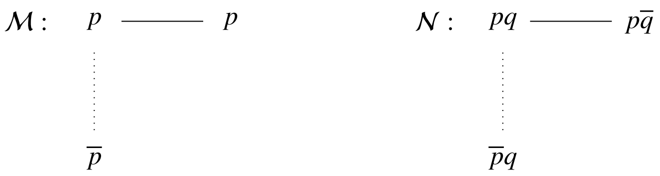
We will use $\mathcal{M}, w \leftleftharpoons \mathcal{N}, v$ to indicate that there is a left-simulation that connects w and v, and $\mathcal{M} \leftleftharpoons \mathcal{N}$ to indicate that there is a *total* left-simulation between $\mathcal{M}$ and $\mathcal{N}$: there is a left-simulation R that links *every* world in $\mathcal{N}$ to *some* world in $\mathcal{M}$.[1] We also write $\mathcal{N} \rightrightharpoons \mathcal{M}$ for $\mathcal{M} \leftleftharpoons \mathcal{N}$.

Theorem 5. — *If $\mathcal{M} \leq \mathcal{N}$ then $\mathcal{M} \leftleftharpoons \mathcal{N}$.*

Proof. — Assume $\mathcal{M} \leq \mathcal{N}$. Then there is a restricted model $\mathcal{K}$ with $\mathcal{M} \oplus \mathcal{K} \leftrightarrows \mathcal{N}$. Let Z be a total bisimulation between $W^{\mathcal{M}\oplus\mathcal{K}}$ and $W^{\mathcal{N}}$. Define C as wCv iff there is some world $k \in W^{\mathcal{K}}$ with $(w,k)Zv$. C is easily seen to be a total left-simulation between $\mathcal{M}$ and $\mathcal{N}$. The restricted invariance property follows from the definition of the valuation on $\mathcal{M} \oplus \mathcal{K}$. The zag property follows from the definition of the accessibility relations on $\mathcal{M} \oplus \mathcal{K}$. Thus, $\mathcal{M} \leftleftharpoons \mathcal{N}$. ■

Note that the converse of Theorem 5 does not hold without restrictions. For example, let $\mathcal{M}$ and $\mathcal{N}$ be the following two S5 models:

1. Note that the *totality* here is different from the totality of bisimulation.



It is clear that $\mathcal{M} \underline{\leftarrow} \mathcal{N}$. Now suppose towards a contradiction that there exists a $\mathcal{K}$ with $\mathcal{M} \oplus \mathcal{K} \underline{\leftrightarrow} \mathcal{N}$. Since there is a pq world in $\mathcal{N}$ there must be a world v in $\mathcal{K}$ with q true in v and v compatible with a p world. So then v is compatible with the top right world in the picture of $\mathcal{M}$. Call this world w. Then (w, v) should be in $W^{\mathcal{M}\oplus\mathcal{K}}$ with $V^{\mathcal{M}\oplus\mathcal{K}}((w, v)) = \{p, q\}$. But since w is not related to a $\overline{p}$ world in one step, by the definition of composition (w, v) should also not be related to a $\overline{p}$ world in one step. So (w, v) cannot be bisimilar to any world in $\mathcal{N}$.

We will prove the converse of Theorem 5 for propositionally differentiated models.

Remark 6. — Van Ditmarsch and French (van Ditmarsch *et al.*, 2009) prove that for (non-restricted) standard epistemic models $\mathcal{M}$, $\mathcal{N}$: $\mathcal{M} \underline{\leftarrow} \mathcal{N}$ iff there is an action model A with $\mathcal{M} \otimes A \underline{\leftrightarrow} \mathcal{N}$ where $\otimes$ is the standard product update in DEL. The operation of updating with an action model is more powerful than that of composing with another epistemic model, because of our restriction to Boolean agreement only. We would move closer to the 'left-similarity test' of van Ditmarsch en French if we would strengthen the agreement requirements on composition. See previous Remark 2. □

Definition 7. — *Call a model $\mathcal{M}$* propositionally differentiated *if it holds for all worlds w, w' of $\mathcal{M}$ that if w and w' have the same valuation then $w \underline{\leftrightarrow} w'$.*

For example, the models for Muddy Children scenario and the Russian Cards Problem discussed in (van Ditmarsch, 2003) are indeed propositionally differentiated. Restricted to this simple but useful class of models we have the exact correspondence of $\underline{\leftarrow}$ and $\leq$:

Theorem 8. — *Let $\mathcal{M}$ be a propositionally differentiated model. Then $\mathcal{M} \underline{\leftarrow} \mathcal{N}$ implies $\mathcal{M} \leq \mathcal{N}$.*

Proof. — We assume without loss of generality that $\mathcal{M}$ is 'bisimulation minimal' (for each model we can consider its bisimulation contraction such that no different states are bisimilar). This, together with the fact that $\mathcal{M}$ is propositionally differentiated, gives that different worlds of $\mathcal{M}$ have different valuations. Let C be a total left-simulation between $\mathcal{M}$ and $\mathcal{N}$. Note that since C is total and $\mathcal{M}$ is propositionally differentiated and bisimulation minimal, for each $v \in W^{\mathcal{N}}$ there is exactly one $w \in W^{\mathcal{M}}$ such that $V^{\mathcal{M}}(w) = V^{\mathcal{N}}(v) \cap Q^{\mathcal{M}}$ and wCv.

We will show that $\mathcal{M} \oplus \mathcal{N} \underline{\leftrightarrow} \mathcal{N}$. Let the relation Z between $\mathcal{M} \oplus \mathcal{N}$ and $\mathcal{N}$ be defined as

$$(w, v)Zv' \text{ iff } v = v'$$

We claim Z is a total bisimulation. Note that it follows from the fact that C is total and the definition of Z that Z is total.

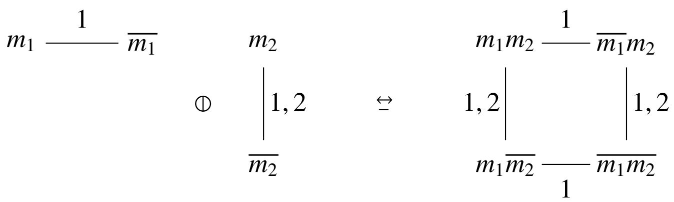
Suppose $(w, v)Zv$. By construction of $\mathcal{M} \oplus \mathcal{N}$, $V^{\mathcal{M}\oplus\mathcal{N}}((w, v)) = V^{\mathcal{M}}(w) \cup V^{\mathcal{N}}(v) = V^{\mathcal{N}}(v)$. This proves the invariance property.

Suppose $(w, v)Zv$ and $(w, v)R_i^{\mathcal{M}\oplus\mathcal{N}}(w', v')$. By construction of $\mathcal{M} \oplus \mathcal{N}$ this means $wR_i^{\mathcal{M}}w'$ and $vR_i^{\mathcal{N}}v'$. By construction of Z, $(w', v')Zv'$. This proves the zig property.

Suppose $(w, v)Zv$ and $vR_i^{\mathcal{N}}v'$. Since $(w, v) \in W^{\mathcal{M}\oplus\mathcal{N}}$, $V^{\mathcal{M}}(w) = V^{\mathcal{N}}(v) \cap Q^{\mathcal{M}}$. So then w is the unique element of $W^{\mathcal{M}}$ that has that property for v, and wCv. Then because C is a left-simulation, there must be some w' such that $wR_i^{\mathcal{M}}w'$ and $w'Cv'$. Since $w'Cv'$, $V^{\mathcal{M}}(w') = V^{\mathcal{N}}(v') \cap Q^{\mathcal{M}}$ so $(w', v') \in W^{\mathcal{M}\oplus\mathcal{N}}$. Since $wR_i^{\mathcal{M}}w'$ and $vR_i^{\mathcal{N}}v'$, $(w, v)R_i^{\mathcal{M}\oplus\mathcal{N}}(w', v')$ and by definition of Z, $(w', v')Zv'$. This proves the zag property. ■

Now we consider a particular type of composition: the *expansion* of a model with an extra vocabulary. Let Q^I be the *universal ignorance* model for Q, i.e. $Q^I = (W, I, R, V, Q)$ with $W = \mathcal{P}(Q)$, $R_i = W^2$, and V being the identity function. If $\mathcal{M} = (W, I, R, V, Q)$ is a restricted model and Q_1 is a set of propositional letters, then we define the expanded model for the larger vocabulary $Q \cup Q_1$ as follows: $\mathcal{M} \triangleleft Q_1 = \mathcal{M} \oplus {Q_1}^I$.

Here is an example of expanding with a single new propositional letter m_2. Note that here and henceforth, we omit the i-relation between the worlds that are linked by a path of i relations of arbitrary length in the picture.



Model expansion to a larger vocabulary will be used in the definition of action model update for restricted models, in Section 3. Expansions with respect to different vocabularies are bisimilar to each other, as long as the expanded vocabulary is the same:

Proposition 9. — *For any model $\mathcal{M}$, and vocabularies $X, Y \subseteq P$ of propositional letters, if $X \cup Q^{\mathcal{M}} = Y \cup Q^{\mathcal{M}}$ then $\mathcal{M} \triangleleft X \leftrightarrows \mathcal{M} \triangleleft Y$.*

Proof. — Let relation $Z \subseteq W^{\mathcal{M}\triangleleft X} \times W^{\mathcal{M}\triangleleft Y}$ be given by:

$$(w, X')Z(w', Y') \iff w = w' \text{ and } V^{\mathcal{M}}(w) \cup X' = V^{\mathcal{M}}(w') \cup Y'$$

We claim that Z is a total bisimulation. Totality follows from the fact that $X \cup Q^{\mathcal{M}} = Y \cup Q^{\mathcal{M}}$. Now we check the three conditions of bisimulation. Suppose $(w, X')Z(w, Y')$ then by definition of Z, $V^{\mathcal{M}}(w) \cup X' = V^{\mathcal{M}}(w) \cup Y'$, namely the invariance condition holds. Then based on totality, it is easy to show the Zig and Zag conditions also hold. ■

Also the expansion is *monotonic* in the sense that the expansion with a larger extra vocabulary is restricted bisimilar to the expansion with a smaller extra vocabulary:

Proposition 10. — *For any model $\mathcal{M}$, any vocabularies X, Y such that $Y \subseteq X$, if $X \cap Q^{\mathcal{M}} = \emptyset$ then $\mathcal{M} \triangleleft X \leftrightarrows_{Q^{\mathcal{M}} \cup Y} \mathcal{M} \triangleleft Y$.*

PROOF. — Let relation $Z \subseteq W^{\mathcal{M}\triangleleft X} \times W^{\mathcal{M}\triangleleft Y}$ be given by:

$$(s, X')Z(s', Y') \iff s = s' \text{ and } Y' = X' \cap Y$$

It is not hard to verify that Z is a total bisimulation restricted to the vocabulary $Q^{\mathcal{M}} \cup Y$. ■

If $\mathcal{M}$ is left-similar to $\mathcal{N}$ then the expansion of $\mathcal{M}$ with $Q^{\mathcal{N}}$ is also left-similar to $\mathcal{N}$.

PROPOSITION 11. — *If $\mathcal{M} \leftleftharpoons \mathcal{N}$ then $\mathcal{M} \triangleleft Q^{\mathcal{N}} \leftleftharpoons \mathcal{N}$*

PROOF. — Let Z be a left-simulation which witnesses $\mathcal{M} \leftleftharpoons \mathcal{N}$. Let

$$Z' = \{((w, V^{\mathcal{N}}(v)), v) \mid (w, v) \in Z\}$$

Note that when $(w, v) \in Z$, $(w, V^{\mathcal{N}}(v))$ is indeed in the model $\mathcal{M} \triangleleft Q^{\mathcal{N}}$ due to the restricted invariance condition of Z. Thus Z' is well-defined. Totality follows from the totality of Z. We claim that Z' is a left-simulation between $\mathcal{M} \triangleleft Q^{\mathcal{N}}, (w, V^{\mathcal{N}}(v))$ and $\mathcal{N}, v$. The condition of restricted invariance is obvious. For the Zag condition, suppose $v \xrightarrow{i}_{\mathcal{N}} v'$ then there is a w' such that $w \xrightarrow{i}_{\mathcal{M}} w'$ and $w'Zv'$. Since $\mathcal{M} \triangleleft Q^{\mathcal{N}} = \mathcal{M} \oplus Q^{I}_{\mathcal{N}}$, we have $(w, V^{\mathcal{N}}(v)) \xrightarrow{i}_{\mathcal{M}\triangleleft Q^{\mathcal{N}}} (w', V^{\mathcal{N}}(v'))$ and $(w', V^{\mathcal{N}}(v'))Z'v'$. ■

Let p range over P and i over I, the language of *Propositional Dynamic Logic* (PDL) over P, I, notation L_{PI}, is given by:

$$\begin{aligned} \phi \quad &::= \quad \top \mid p \mid \neg\phi \mid \phi \vee \phi \mid \langle\alpha\rangle\phi \\ \alpha \quad &::= \quad i \mid ?\phi \mid \alpha;\alpha \mid \alpha \cup \alpha \mid \alpha^* \end{aligned}$$

We employ the usual abbreviations. In particular $\bot$ abbreviates $\neg\top$. The satisfaction relation for $L_{P,I}$ formulas on pointed models $\mathcal{M}, w$ ($\mathcal{M} \vDash_w \phi$) is defined as in usual PDL semantics (cf. e.g., (Harel *et al.*, 2000)). Since PDL is bisimulation invariant, as a straightforward consequence of Proposition 10, we have:

PROPOSITION 12. — *For any model $\mathcal{M}$, if $X \cap Q^{\mathcal{M}} = \emptyset$ and $Y \subseteq X$ then for any $\phi \in L_{Q\cup Y,I}$: $\mathcal{M} \triangleleft X \vDash_{(s,X')} \phi \iff \mathcal{M} \triangleleft Y \vDash_{(s,X'\cap Y)} \phi$.*

We will use this proposition to prove Theorem 19 in Section 3.

The diamond fragment of L_{PI} is given by the formulas of the syntactic form of ϕ in the following definition:

$$\begin{aligned} \phi \quad &::= \quad \psi \mid \langle\alpha\rangle\phi \mid \phi \vee \phi \mid \phi \wedge \phi. \\ \psi \quad &::= \quad \top \mid p \mid \neg\psi \mid \psi \vee \psi \\ \alpha \quad &::= \quad i \mid ?\phi \mid \alpha;\alpha \mid \alpha \cup \alpha \mid \alpha^* \end{aligned}$$

It is well-known that diamond formulas are preserved under left simulation (cf. e.g., (Blackburn *et al.*, 2001)). The following theorem generalizes this to cases where the vocabularies of the two models may be different.

THEOREM 13. — *If $\mathcal{M}, w \leftharpoondown \mathcal{N}, v$ then all formulas ϕ over the vocabulary $Q^{\mathcal{M}}$ in the diamond fragment of L_{PI} are preserved from right to left under left simulation: if $\mathcal{N} \models_v \phi$ then $\mathcal{M} \models_w \phi$.*

PROOF. — Let C be a left simulation with wCv. We prove the property by induction on the construction of ϕ. If ϕ has the form ψ and is a propositional letter p, then p is in the vocabulary of $\mathcal{M}$, and the result holds by the restricted invariance property of C. Purely Boolean combinations of ϕ are obvious. So the property holds for all Boolean formulas ψ. As an example of the reasoning for $\langle\alpha\rangle$ we give the case of $\langle i\rangle\phi$. Suppose the property holds for ϕ, and assume $\mathcal{N} \models_v \langle i\rangle\phi$. Then there is a v' with $v \xrightarrow{i} v'$ and $\mathcal{N} \models_{v'} \phi$. By the zag property of C, there is a w' with $w \xrightarrow{i} w'$ and $w'Cv'$. By the induction hypothesis, $\mathcal{M} \models_{w'} \phi$, and therefore $\mathcal{M} \models_w \langle i\rangle\phi$. ■

Note that it follows from this Theorem (together with Theorem 5) that positive knowledge in a component model $\mathcal{M}$ is preserved in $\mathcal{M} \oplus \mathcal{N}$: knowledge (box) formulas are preserved in the other direction from diamond formulas. This shows that we can see the operation of model composition as an operation of incorporation of new knowledge in an existing knowledge base, or as an operation of combining knowledge bases into a new knowledge base.

We can get rid of the vocabulary constraint. From Proposition 11, $\mathcal{M}, w \leftharpoondown \mathcal{N}, v$ implies

$$\mathcal{M} \triangleleft Q^{\mathcal{N}}, (w, V(v)) \leftharpoondown \mathcal{N}, v,$$

thus we have:

COROLLARY 14. — *If $\mathcal{M}, w \leftharpoondown \mathcal{N}, v$ then all formulas ϕ in the diamond fragment of L_{PI} are preserved from right to left under left simulation: if $\mathcal{N} \models_v \phi$ then $\mathcal{M} \triangleleft Q^{\mathcal{N}} \models_{(w,V(v))} \phi$.*

The box fragment of L_{PI} is defined dually to the diamond fragment. Box formulas are preserved in the other direction:

THEOREM 15. — *If $\mathcal{M}, w \leftharpoondown \mathcal{N}, v$ then all formulas ϕ over the vocabulary $Q^{\mathcal{M}}$ in the box fragment of L_{PI} are preserved from left to right under left simulation: if $\mathcal{M} \models_w \phi$ then $\mathcal{N} \models_v \phi$.*

As the above theorems suggest, our composition approach hints at a way of epistemic model checking by looking at component models only. The following theorem gives another example where properties of a large model can be checked by looking at its components.

THEOREM 16 (PRESERVATION). — *Suppose $\mathcal{M} = \mathcal{M}_0 \oplus \cdots \oplus \mathcal{M}_n$. If the epistemic models $\mathcal{M}_0, \ldots, \mathcal{M}_n$ have pairwise disjoint vocabularies $Q^0, Q^1, \ldots, Q^n$, then for any L_{PI} formula ϕ based on Q^i and any $w = \langle w_0, \ldots, w_n\rangle \in W^{\mathcal{M}}$: $\mathcal{M}_i \vDash_{w_i} \phi \iff \mathcal{M} \vDash_w \phi$.*

PROOF. — Let Z_i be the relation between the worlds of $\mathcal{M}$ and the worlds of $\mathcal{M}_i$ given by $\overline{v}Z_iv$ iff $\overline{v}[i] = v$ (where $\overline{v}[i]$ is the ith component of the vector $\overline{v}$). We claim that Z_i is a Q^i-restricted bisimulation (a bisimulation with the invariance condition restricted to Q^i). Since L_{PI} is invariant under bisimulation then the preservation result holds. Now we prove the claim. Assume $\overline{v}Z_iv$. Then $V^{\mathcal{M}}(\overline{v}) \cap Q^i = V^{\mathcal{M}_i}(v)$, by the definition of parallel composition and the fact that Q^i is disjoint from the other vocabularies. Thus, Q^i-restricted invariance holds. Next suppose $\overline{v}R_k\overline{u}$. Then by the definition of the accessibility relations on $\mathcal{M}$, $\overline{v}[i]R_k\overline{u}[i]$, whence, by definition of Z_i, there is a u in the domain of $\mathcal{M}_i$ with $\overline{u}Z_iv$. It follows that the Zig condition holds. Finally, assume there is a u in the domain of $\mathcal{M}_i$ with vR_ku. Consider the state $\overline{u}$ given by $\overline{u}[i] = u$ and

$\overline{u}[j] = \overline{v}[j]$ for $j \neq i$. Then $\overline{u}$ is in the domain of $\mathcal{M}$, because of the fact that the vocabulary Q_i is disjoint with each of the other vocabularies Q_j. By reflexivity of the component models and the fact that $vR_k u$, $\overline{v}R_i\overline{u}$. From the definitions of $\overline{u}$ and Z_i we get that $\overline{u}Z_i v$, i.e., the Zag condition holds. ■

For an example application, consider a muddy children model of 2^n components. As we demonstrated earlier, this can be viewed as built from n two-component models, each with its own vocabulary for talking about the muddiness of a single child. Any epistemic statement that talks about the muddiness of a single child in the big model can be checked by evaluation in a single two-world component.

At this point, a natural question arises: when can an epistemic model be composed by other (smaller) models? We will come back to this question in Section 4.

3. Composing action models

In the framework of dynamic epistemic logic, an *action model* is like an epistemic model, but with valuations replaced by precondition formulas taken from an appropriate language. Here we generalize the standard definition of action models to action models with arbitrary (partial) vocabularies.

A *restricted action model* over a proposition set P for an agent set I is a quintuple (E, I, S, T, Q) where Q is a subset of P, E is a set of events, S is a function that assigns to every $i \in I$ a binary relation S_i on E and T is a function that assigns to every $e \in E$ an $L_{P,I}$ formula over Q, the so-called *precondition* of event e. We will use $E^{\mathcal{A}}$, $S^{\mathcal{A}}$, $T^{\mathcal{A}}$ and $Q^{\mathcal{A}}$ to denote the elements of an action model $\mathcal{A}$. We also write $e \xrightarrow{i} e'$ for $eS_i e'$ in an action model.

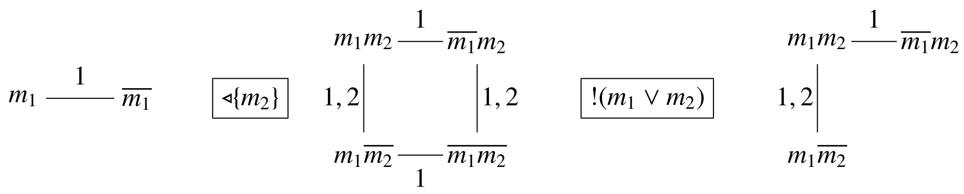
We will now give a version of action model update $\otimes$ (see (Baltag *et al.*, 1999)) for restricted models. Model expansion to a larger vocabulary is used in the definition of product update to ensure that the resulting model has a vocabulary consisting of the union of the vocabulary of the static model and the vocabulary of the action model.

Let $\mathcal{M}$ be a restricted epistemic model and $\mathcal{A}$ be a restricted action model for the same agent set I. Let X be the new vocabulary, i.e., $X = Q^{\mathcal{A}} - Q^{\mathcal{M}}$. The *extended product update* $\mathcal{M} \circledcirc \mathcal{A}$ is defined as $(\mathcal{M} \triangleleft X) \otimes (E^{\mathcal{A}}, I, S^{\mathcal{A}}, T^{\mathcal{A}})$ where $\otimes$ is the usual update product. Namely $\mathcal{M} \circledcirc \mathcal{A} = (W', I, R', V', Q')$ where:

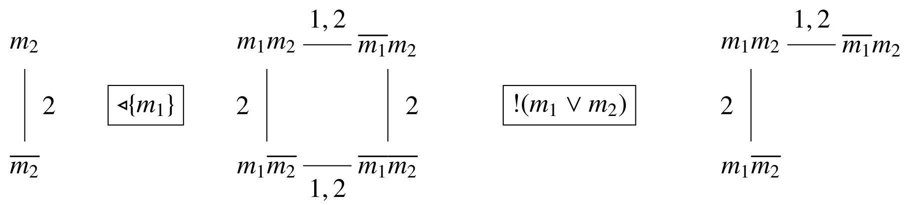
1) $W' = \{(w, Y, e) \mid w \in W^{\mathcal{M}}, e \in E^{\mathcal{A}}, Y \subseteq X, M \triangleleft X \models_{(w,Y)} T^{\mathcal{A}}(e)\}$,
2) $(w, Y, e)R'_i(w', Y', e')$ iff $wR_i^{\mathcal{M}}w'$ and $eS_i^{\mathcal{A}}e'$,
3) $V'(w, Y, e) = V^{\mathcal{M}}(w) \cup Y$.
4) $Q' = Q^{\mathcal{M}} \cup X = Q^{\mathcal{M}} \cup Q^{\mathcal{A}}$.

According to this definition, to update a restricted action model on a restricted epistemic model, we first need to expand the vocabulary of the restricted epistemic model to incorporate the propositional letters relevant in $\mathcal{A}$, and then perform a usual product update.

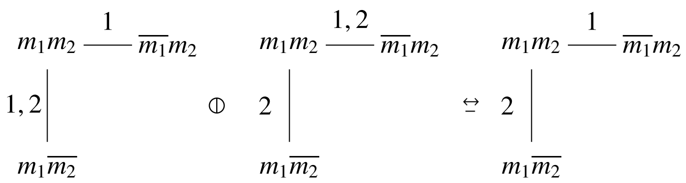
Here is an example of an update with a public announcement 'At least one of the two children is muddy' (denoted as $!(m_1 \vee m_2)$). This is represented by an action model with a single event having precondition $m_1 \vee m_2$. Note that, according to our definition, the update involves model expansion, so it is essentially a two-step process:



Here is an update of the other component of the two-muddy-children model:



And here is the outcome of composing the two update results:



This is the same as the result of public announcement of $m_1 \vee m_2$ ("at least one of you is muddy") on the composition on $m_1 \overset{1}{\longleftrightarrow} \overline{m_1}$ and $m_2 \overset{2}{\longleftrightarrow} \overline{m_2}$. The following theorem expresses that this outcome is not accidental: update of a composed model yields the same result (modulo bisimulation) as composing the updates of its components, provided the action model satisfies certain conditions. We call a restricted action model $\mathcal{A}$ *propositionally differentiated* if the preconditions are purely Boolean formulas and any two states in $\mathcal{A}$ have mutually exclusive preconditions.

LEMMA 17. — *Let* $\mathcal{M}_1 = (\mathcal{M} \oplus \mathcal{N}) \circledcirc \mathcal{A}$ *and* $\mathcal{M}_2 = (\mathcal{M} \circledcirc \mathcal{A}) \oplus (\mathcal{N} \circledcirc \mathcal{A})$*, and let relation* $Z \subseteq W^{\mathcal{M}_1} \times W^{\mathcal{M}_2}$ *be given by:*

$$((s,t,X),e)Z((s',X_1,e'),(t',X_2,e'')) \text{ iff } s = s', t = t', e = e' = e'' \text{ and } X = X_1 \cap X_2.$$

Then Z is total.

PROOF. — The (long) proof of this lemma can be found in Appendix A. ■

THEOREM 18. — *If A is propositionally differentiated then:*

$$(\mathcal{M} \oplus \mathcal{N}) \circledcirc \mathcal{A} \leftrightarrows (\mathcal{M} \circledcirc \mathcal{A}) \oplus (\mathcal{N} \circledcirc \mathcal{A}).$$

PROOF. — Let $\mathcal{M}_1 = (\mathcal{M} \oplus \mathcal{N}) \circledcirc \mathcal{A}$ and $\mathcal{M}_2 = (\mathcal{M} \circledcirc \mathcal{A}) \oplus (\mathcal{N} \circledcirc \mathcal{A})$. Let relation $Z \subseteq W^{\mathcal{M}_1} \times W^{\mathcal{M}_2}$ be given by:

$$((s, t, X), e)Z((s', X_1, e'), (t', X_2, e'')) \text{ iff } s = s', t = t', e = e' = e'' \text{ and } X = X_1 \cap X_2.$$

We need to show that Z is a total bisimulation between $\mathcal{M}_1$ and $\mathcal{M}_2$. The totality of Z is proved in Lemma 17. Here we focus on the three conditions of bisimulation. Suppose $((s, t, X), e)$ and $((s, X_1, e'), (t, X_2, e''))$ exist in $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively and $((s, t, X), e)Z((s, X_1, e'), (t, X_2, e''))$.

For invariance we need to show

$$V^{\mathcal{M}}(s) \cup V^{\mathcal{N}}(t) \cup (X_1 \cap X_2) = V^{\mathcal{M}}(s) \cup X_1 \cup V^{\mathcal{N}}(t) \cup X_2$$

Since the only difference between the left hand side and right hand side is about $X_1, X_2 \subseteq Q^{\mathcal{A}}$, showing the following suffices:

$$(V^{\mathcal{M}}(s) \cup V^{\mathcal{N}}(t) \cup (X_2 \cap X_2)) \cap Q^{\mathcal{A}} = (V^{\mathcal{M}}(s) \cup X_1 \cup V^{\mathcal{N}}(t) \cup X_2) \cap Q^{\mathcal{A}} \qquad (\star)$$

Since (s, X_1, e) in $\mathcal{M} \circledcirc \mathcal{A}$ and (t, X_2, e) in $\mathcal{N} \circledcirc \mathcal{A}$ are compatible, we have:

$$(V^{\mathcal{M}}(s) \cup X_1) \cap Q^{\mathcal{A}} = (V^{\mathcal{N}}(t) \cup X_2) \cap Q^{\mathcal{A}} \qquad (\star\star)$$

Now let us massage the left hand side of $(\star)$:

$$\begin{aligned}
&(V^{\mathcal{M}}(s) \cup V^{\mathcal{N}}(t) \cup (X_1 \cap X_2)) \cap Q^{\mathcal{A}} \\
&= ((V^{\mathcal{M}}(s) \cup V^{\mathcal{N}}(t) \cup X_1) \cap (V^{\mathcal{M}}(s) \cup V^{\mathcal{N}}(t) \cup X_2)) \cap Q^{\mathcal{A}} \\
&= ((V^{\mathcal{M}}(s) \cup V^{\mathcal{N}}(t) \cup X_1) \cap Q^{\mathcal{A}} \cap (V^{\mathcal{M}}(s) \cup V^{\mathcal{N}}(t) \cup X_2)) \cap Q^{\mathcal{A}} \\
&= (((V^{\mathcal{M}}(s) \cup X_1) \cap Q^{\mathcal{A}}) \cup (V^{\mathcal{N}}(t) \cap Q^{\mathcal{A}})) \cap (((V^{\mathcal{N}}(t) \cup X_2) \cap Q^{\mathcal{A}}) \cup (V^{\mathcal{M}}(s) \cap Q^{\mathcal{A}})) \\
&= (((V^{\mathcal{M}}(s) \cup X_1) \cap Q^{\mathcal{A}}) \cup (V^{\mathcal{N}}(t) \cap Q^{\mathcal{A}})) \cap ((V^{\mathcal{M}}(s) \cup X_1) \cap Q^{\mathcal{A}}) \cup (V^{\mathcal{M}}(s) \cap Q^{\mathcal{A}})) \text{ (by } (\star\star)) \\
&= ((V^{\mathcal{M}}(s) \cup X_1) \cap Q^{\mathcal{A}}) \cup (V^{\mathcal{N}}(t) \cap V^{\mathcal{M}}(s) \cap Q^{\mathcal{A}}) \\
&= ((V^{\mathcal{M}}(s) \cup X_1) \cap Q^{\mathcal{A}}) \qquad (\text{ since } V^{\mathcal{N}}(t) \cap V^{\mathcal{M}}(s) \subseteq V^{\mathcal{M}}(s)) \\
&= ((V^{\mathcal{M}}(s) \cup X_1) \cap Q^{\mathcal{A}}) \cup ((V^{\mathcal{N}}(t) \cup X_2) \cap Q^{\mathcal{A}}) \qquad (\text{ from } (\star\star)) \\
&= (V^{\mathcal{M}}(s) \cup X_1 \cup V^{\mathcal{N}}(t) \cup X_2) \cap Q^{\mathcal{A}}
\end{aligned}$$

This proves the invariance requirement.

Now assume $((s, t, X), e)Z((s, X_1, e_1), (t, X_2, e_2))$ and $((s, t, X), e) \xrightarrow{i} ((s', t', X'), e')$ in $\mathcal{M}_1$, then $e = e_1 = e_2$, $s \xrightarrow{i} s'$ in $\mathcal{M}$, $t \xrightarrow{i} t'$ in $\mathcal{N}$ and $e \xrightarrow{i} e'$ in $\mathcal{A}$. From totality (Lemma 17), in $\mathcal{M}_2$ there exists $((s', X_1', e'), (t', X_2', e'))$ for some X_1' and X_2' such that

$$((s', t', X'), e')Z((s', X_1', e'), (t', X_2', e')).$$

According to the definition of relations in $\mathcal{M}_2$, it is not hard to see that

$$((s, X_1, e), (t, X_2, e)) \xrightarrow{i} ((s', X_1', e'), (t', X_2', e'))$$

This proves the Zig requirement.

Suppose $((s, t, X), e)Z((s, X_1, e_1), (t, X_2, e_2))$ and

$$((s, X_1, e), (t, X_2, e)) \xrightarrow{i} ((s', X_1', e'), (t', X_2', e''))$$

Since $\mathcal{A}$ is propositional differentiated, $e' = e''$. Therefore $s \xrightarrow{i}_{\mathcal{M}_1} s'$, $t \xrightarrow{i}_{\mathcal{M}_2} t'$, and $e \xrightarrow{i}_{\mathcal{A}} e'$. From Lemma 17, in $\mathcal{M}_1$ there exists (s', X', e') for some X' such that:

$$((s', t', X'), e')Z((s', X_1', e'), (t', X_2', e'))$$

It follows that $((s, t, X), e) \xrightarrow{i} ((s', t', X'), e')$. This proves the Zag condition. ■

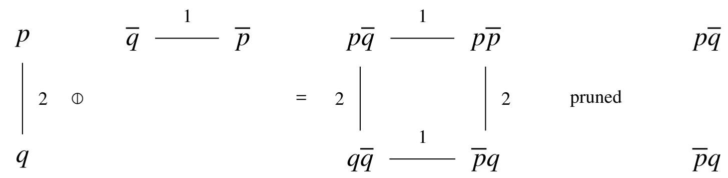
Action models can be viewed as encodings of communicative actions that take place, together with information about how various agents are affected. Public announcement actions affect all agents in the same way, group announcements affect the group that the communication is addressed to in a different way from the outsiders, and so on. These differences are reflected in accessibility relations between actual actions (describing what actually takes place) and alternative actions that some agents may suspect are taking place. Given this, one can imagine that several communicative actions that take place at the same time can be combined in a single parallel composition of actions.

Action models are very similar to epistemic models, and it turns out that composition on action models can be defined in a natural way that agrees with the above intuition.

The *merging composition* $\mathcal{A} \oplus \mathcal{B}$ of two restricted action models $\mathcal{A}$ and $\mathcal{B}$ is given by (E, I, S, T, Q), where:

- $E = \{(w, v) \mid w \in E^{\mathcal{A}}, v \in E^{\mathcal{B}}\}$,
- $(w, v)S_i(w', v')$ iff $wS_i^{\mathcal{A}}w'$ and $vS_i^{\mathcal{B}}v'$
- $T((w, v)) = T^{\mathcal{A}}(w) \wedge T^{\mathcal{B}}(v)$
- $Q = Q^{\mathcal{A}} \cup Q^{\mathcal{B}}$.

A simple example is composing two announcements $!\phi$ and $!\psi$, which results in the announcement of $!(\phi \wedge \psi)$. The composition operator presented here can be viewed as a kind of parallel compositions of events. Consider the following example (where $I = \{1, 2\}$ and the propositions in the picture are preconditions):



The first model captures the event that agent 1 is being told that either p or q is true, while agent 2 can only see it without hearing the exact message. Similarly, the second model reflects the event that 2 is being told either p or q is false without 1 hearing the message. The composition of the two captures that both events are happening at the same time. As we can see, the effect of updating this composed event is the same as updating with an announcement $p \wedge \neg q$ or $\neg p \wedge q$: note that some events in the composed action model may not be executable due to contradictory preconditions thus can be pruned. Intuitively, if agent 1 is told p and he knows that 2 is (truthfully) told either $\neg q$ or $\neg p$ then he actually knows that $\neg q$.

Updating with a composite action model yields the same outcome as updating with its components and then composing the results.

THEOREM 19. — *If $\mathcal{M}$ is propositionally differentiated then $\mathcal{M}\circledcirc(\mathcal{A}\oplus\mathcal{B}) \leftrightarrows (\mathcal{M}\circledcirc\mathcal{A})\oplus(\mathcal{M}\circledcirc\mathcal{B})$.*

PROOF. — Let $\mathcal{M}_1 = \mathcal{M}\circledcirc(\mathcal{A}\oplus\mathcal{B})$ and $\mathcal{M}_2 = (\mathcal{M}\circledcirc\mathcal{A})\oplus(\mathcal{M}\circledcirc\mathcal{B})$. Let the relation $Z \subseteq W^{\mathcal{M}_1} \times W^{\mathcal{M}_2}$ be given by:

$$(s, X, (e, f))Z((s', X_1, e'), (s'', X_2, f')) \text{ iff } V^{\mathcal{M}}(s) = V^{\mathcal{M}}(s') = V^{\mathcal{M}}(s''), e = e', f = f' \text{ and } X = X_1 \cup X_2$$

We first show that Z is total.

$\Rightarrow$: Suppose that $(s, X, (e, f))$ is in $W^{\mathcal{M}_1}$, we will show that there are X_1 and X_2 with $X = X_1 \cup X_2$ such that $((s, X_1, e), (s, X_2, f))$ exists in $W^{\mathcal{M}_2}$. Notice that:

$$\begin{array}{l}
(s, X, (e, f)) \in W^{\mathcal{M}\circledcirc(\mathcal{A}\oplus\mathcal{B})} \\
\iff \mathcal{M} \triangleleft ((Q^{\mathcal{A}} \cup Q^{\mathcal{B}}) - Q^{\mathcal{M}}) \models_{(s,X)} T^{\mathcal{A}}(e) \wedge T^{\mathcal{B}}(f) \\
\iff \mathcal{M} \triangleleft (Q^{\mathcal{A}} - Q^{\mathcal{M}}) \models_{(s,X\cap Q^{\mathcal{A}})} T^{\mathcal{A}}(e) \\
\text{and } \mathcal{M} \triangleleft (Q^{\mathcal{B}} - Q^{\mathcal{M}}) \models_{(s,X\cap Q^{\mathcal{B}})} T^{\mathcal{B}}(f) \qquad \text{(From Proposition 12)}
\end{array}$$

Now let $X_1 = X \cap (Q^{\mathcal{A}} - Q^{\mathcal{M}})$ and $X_2 = X \cap (Q^{\mathcal{B}} - Q^{\mathcal{M}})$. Since $X \subseteq (Q^{\mathcal{A}} \cup Q^{\mathcal{B}}) - Q^{\mathcal{M}}$, $X_1 \cup X_2 = X \cap ((Q^{\mathcal{A}} - Q^{\mathcal{M}}) \cup (Q^{\mathcal{B}} - Q^{\mathcal{M}})) = X$. From the above derivation, $(s, X_1, e) \in W^{\mathcal{M}\circledcirc\mathcal{A}}$ and $(s, X_2, f) \in W^{\mathcal{M}\circledcirc\mathcal{B}}$. Now we show that they are compatible:

$$\begin{array}{rl}
& (V^M(s) \cup X_1) \cap (Q^M \cup Q^B) \\
= & (V^M(s) \cup (X \cap (Q^A - Q^M))) \cap (Q^M \cup Q^B) \\
= & (V^M(s) \cap (Q^M \cup Q^B)) \cup (X \cap (Q^A - Q^M) \cap (Q^M \cup Q^B)) \\
= & V^M(s) \cup (X \cap (Q^A - Q^M) \cap (Q^M \cup Q^B)) \\
= & V^M(s) \cup (X \cap (Q^A - Q^M) \cap Q^B) \\
= & V^M(s) \cup (X \cap (Q^B - Q^M) \cap Q^A) \\
= & V^M(s) \cup (X \cap (Q^B - Q^M) \cap (Q^M \cup Q^A)) \\
= & (V^M(s) \cap (Q^M \cup Q^A)) \cup (X \cap (Q^B - Q^M) \cap (Q^M \cup Q^A)) \\
= & (V^M(s) \cup (X \cap (Q^B - Q^M))) \cap (Q^M \cup Q^A) \\
= & (V^M(s) \cup X_2) \cap (Q^M \cup Q^A)
\end{array}$$

$$
\begin{aligned}
&(V^{\mathcal{M}}(s) \cup X_1) \cap (Q^{\mathcal{M}} \cup Q^{\mathcal{B}})\\
\iff\ & (V^{\mathcal{M}}(s) \cap (Q^{\mathcal{M}} \cup Q^{\mathcal{B}})) \cup (X_1 \cap (Q^{\mathcal{M}} \cup Q^{\mathcal{B}}))\\
\iff\ & V^{\mathcal{M}}(s) \cup (X \cap Q^{\mathcal{A}} \cap (Q^{\mathcal{M}} \cup Q^{\mathcal{B}}))\\
\iff\ & V^{\mathcal{M}}(s) \cup (X \cap Q^{\mathcal{A}} \cap Q^{\mathcal{B}}) \qquad (\text{ since } X \subseteq Q^{\mathcal{A}} \cup Q^{B} - Q^{\mathcal{M}})\\
\iff\ & V^{\mathcal{M}}(s) \cup (X \cap Q^{\mathcal{B}} \cap (Q^{\mathcal{M}} \cup Q^{\mathcal{A}}))\\
\iff\ & (V^{\mathcal{M}}(s) \cup X_2) \cap (Q^{\mathcal{M}} \cup Q^{\mathcal{A}})
\end{aligned}
$$

Therefore $((s, X_1, e), (s, X_2, f))$ exists in $\mathcal{M}_2$.

$\Leftarrow$: Suppose $((s_1, X_1, e), (s_2, X_2, f))$ exists in $W^{\mathcal{M}_2}$. Clearly $V^{\mathcal{M}}(s_1) = V^{\mathcal{M}}(s_2)$. Let $X = X_1 \cup X_2$, we have $X \subseteq (Q^{\mathcal{A}} \cup Q^{\mathcal{B}}) - Q^{\mathcal{M}}$. It is not hard to see that s_1 is compatible with X. To show that $(s_1, X, (e, f)) \in W^{\mathcal{M}_1}$, we only need to check that (s_1, X) satisfies $T(e) \wedge T(f)$. From the existence of $((s_1, X_1, e), (s_2, X_2, f))$ in $\mathcal{M}_2$, we have

$$\mathcal{M} \triangleleft (Q^{\mathcal{A}} - Q^{\mathcal{M}}) \models_{(s,X_1)} T^{\mathcal{A}}(e) \text{ and } \mathcal{M} \triangleleft (Q^{\mathcal{B}} - Q^{\mathcal{M}}) \models_{(s_2,X_2)} T^{\mathcal{B}}(f) \qquad (\star)$$

Note that $(X_1 \cup X_2) \cap (Q^{\mathcal{A}} - Q^{\mathcal{M}}) = X_1$, for otherwise $((s_1, X_1, e), (s_2, X_2, f))$ is not in $W^{|M_2}$. Similarly we have $(X_1 \cup X_2) \cap (Q^{\mathcal{B}} - Q^{\mathcal{M}}) = X_2$. Since $Q^{\mathcal{A}} - Q^{\mathcal{M}} \subseteq Q^{\mathcal{A}} \cup Q^{\mathcal{B}} - Q^{\mathcal{M}}$ and $Q^{\mathcal{B}} - Q^{\mathcal{M}} \subseteq Q^{\mathcal{A}} \cup Q^{\mathcal{B}} - Q^{\mathcal{M}}$, rewriting $(\star)$ we have:

$$
\begin{aligned}
&\mathcal{M} \triangleleft ((Q^{\mathcal{A}} \cup Q^{\mathcal{B}} - Q^{\mathcal{M}}) \cap (Q^{\mathcal{A}} - Q^{\mathcal{M}})) \models_{(s_1, X \cap (Q^{\mathcal{A}} - Q^{\mathcal{M}}))} T^{\mathcal{A}}(e) \text{ and}\\
&\mathcal{M} \triangleleft ((Q^{\mathcal{A}} \cup Q^{\mathcal{B}} - Q^{\mathcal{M}}) \cap (Q^{\mathcal{B}} - Q^{\mathcal{M}})) \models_{(s_2, X \cap (Q^{\mathcal{B}} - Q^{\mathcal{M}}))} T^{\mathcal{B}}(f)
\end{aligned}
$$

Since $V^{\mathcal{M}}(s_1) = V^{\mathcal{M}}(s_2)$ and $\mathcal{M}$ is propositionally differentiated, $\mathcal{M}, s_1 \leftrightarroweq \mathcal{M}, s_2$. Now from Proposition 12 it is not hard to see that:

$$\mathcal{M} \triangleleft (Q^{\mathcal{A}} \cup Q^{\mathcal{B}} - Q^{\mathcal{M}}) \models_{(s_1,X)} T^{\mathcal{A}}(e) \wedge T^{\mathcal{B}}(f)$$

This proves that $((s_1, X, (e, f))$ exists in $\mathcal{M}_1$.

The invariance condition of Z is straightforward since $X = X_1 \cup X_2$ and $V^{\mathcal{M}}(s_1) = V^{\mathcal{M}}(s_2) = V^{\mathcal{M}}(s)$. Based on the totality of Z and the fact that $((s, X, (e, f))Z((s_1, X_1, e'), (s_2, X_2, f')$ implies $V^{\mathcal{M}}(s) = V^{\mathcal{M}}(s_1) = V^{\mathcal{M}}(s_2)$, which in turn implies $\mathcal{M}, s \leftrightarroweq \mathcal{M}, s_1 \leftrightarroweq \mathcal{M}, s_2$, the Zig and Zag properties can be verified easily. ■

4. Decomposition

In this section, we address the question of decomposition: what kind of model can be decomposed into what kind of form? We will mainly look at a particular class of models which is useful in multi-agent systems. In the interpreted systems literature (cf. e.g., (Engelhardt *et al.*, 1998)), a basic proposition $p \in Q$ is *i-local* for $i \in I$ in a model $\mathcal{M}$, if for any w, v in $W^{\mathcal{M}}$: wR_iv implies that $(p \in V^{\mathcal{M}}(w) \iff p \in V^{\mathcal{M}}(v))$. Intuitively, the i-local propositions are the *atomic observables* of agent i and thus agent i also knows whether they are true. Here we extend this idea by considering not only basic propositions but also their Boolean combinations. We say

$\mathcal{M}$ is *locally generated* if, for every agent i, there is a non-empty set of Boolean formulas Φ_i (the set of local observables) based on $Q^{\mathcal{M}}$ such that:

$$\text{for all } w, w' \in W^{\mathcal{M}}, wR_iw' \text{ iff for all } \varphi \in \Phi_i, \mathcal{M} \models_w \varphi \Leftrightarrow \mathcal{M} \models_{w'} \varphi$$

Intuitively, a model is locally generated if those local observables determine the epistemic relations in the model. The Muddy Children model is a typical example of a locally generated model (the set of local observables for i is $\{m_j \mid j \neq i, j \in I\}$). As the following two propositions will show, locally generated models are essentially propositionally differentiated models, which we considered in Theorem 8.

Proposition 20. — *A locally generated model is bisimilar to a propositionally differentiated model. More precisely, its bisimulation contraction is propositionally differentiated.*

Proof. — Given a locally generated model $\mathcal{M}$, suppose Φ_i is the set of local observables for i. Let $Z = \{(w, v) \mid V^{\mathcal{M}}(w) = V^{\mathcal{M}}(v)\}$. We show Z is a bisimulation. Assume wZv. The invariance condition is trivial. For Zig, suppose $w \xrightarrow{i} w'$. Since $\mathcal{M}$ is locally generated, for any $\phi \in \Phi_i : \mathcal{M} \vDash_w \phi \iff \mathcal{M} \vDash_{w'} \phi$. Since Φ_i contains only Boolean formulas and $V^{\mathcal{M}}(w) = V^{\mathcal{M}}(v)$, we have for any $\phi \in \Phi_i : \mathcal{M} \vDash_v \phi \iff \mathcal{M} \vDash_{w'} \phi$. Again due to the definition of the relations in a locally generated model, we have vR_iw'. Obviously $w'Zw'$, thus it proves the Zig condition. The same argument works for the Zag condition. Therefore it is easy to see that the bisimulation contraction of $\mathcal{M}$ is propositionally differentiated. ■

On the other hand, we also have:

Proposition 21. — *If the set of propositional letters is finite then a propositionally differentiated model is bisimilar to a locally generated model. More precisely, its bisimulation contraction is locally generated.*

Proof. — Suppose $\mathcal{M}$ is propositionally differentiated. Let $\mathcal{N}$ be its bisimulation contraction. Clearly $\mathcal{N}$ is also propositionally differentiated. Let $|W^{\mathcal{N}}|_{R_i}$ be the partitioning of $W^{\mathcal{N}}$ according to the equivalence relation R_i. Since $\mathcal{N}$ is a propositionally differentiated bisimulation contraction and there are finitely manly propositional letters, we can characterize each world by a conjunction of literals. Then we can characterize each equivalence class in $|W^{\mathcal{N}}|_{R_i}$ by a disjunction of these characterizations. Let Φ_i be the set of these disjunctions. It is not hard to see that $\mathcal{M}$ is locally generated from these Φ_i. ■

We can decompose a locally generated model into certain components in an intuitive way.

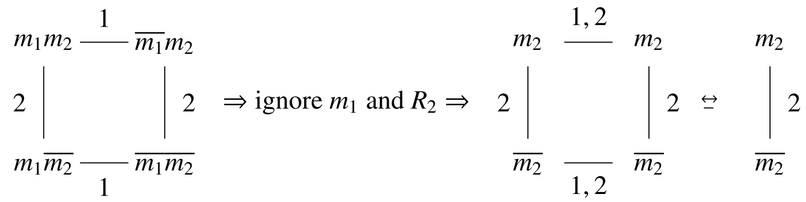
Theorem 22 (Decomposition by agents). — *Given a set of agents $I = \{1, 2, \ldots n\}$. If $\mathcal{M} = (W, I, R, V, Q)$ is locally generated w.r.t. $\Phi_1, \ldots, \Phi_n$, then there are models $\mathcal{M}_1, \ldots, \mathcal{M}_n$ and $\mathcal{M}_0$ such that:*

- $\mathcal{M} \leftrightarroweq (\mathcal{M}_0 \oplus \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_n)$;
- $|W^{\mathcal{M}_j}| \leq |W|$ *and* $\mathcal{M}_i$ *is bisimulation contracted;*
- $Q^{\mathcal{M}_j} = \{p \in Q^{\mathcal{M}} \mid p$ *appears in* $\Phi_j\}$ *for* $j > 0$;

PROOF. — Let $Q^i = \{p \in Q^{\mathcal{M}} \mid p$ appears in $\Phi_i\}$ and $Q^0 = Q^{\mathcal{M}}$ then we define $\mathcal{N}_i = (W^{\mathcal{M}}, I, R^{\mathcal{N}_i}, V_i, Q^i)$ where V_i is the restriction of $V^{\mathcal{M}}$ to Q^i and

$$w \xrightarrow{j}_{\mathcal{N}_i} w' \iff \begin{cases} w \xrightarrow{j}_{\mathcal{M}} w' \text{ if } j = i \\ \text{always} \quad \text{if } i \neq j \end{cases}$$

Intuitively, for each $i \in I$: $\mathcal{N}_i$ is a 'local' model for agent i obtained by ignoring the non-local information: atomic propositions not mentioned in the i-observables and epistemic accessibility relations for agents other than i. Note that by ignoring the epistemic relations for j we mean setting R_j to be universal. For example:



By our definition, the relations in $\mathcal{N}_0$ are universal. Intuitively, $\mathcal{N}_0$ captures all the possible *states of affairs* in $\mathcal{M}$. Let a relation $Z \subseteq W^{\mathcal{M}} \times W^{\mathcal{N}_0 \oplus \cdots \oplus \mathcal{N}_n}$ be given as follows:

$$wZ(w_0, w_1, \ldots, w_n) \iff w = w_0$$

Now let us verify that Z is indeed a total bisimulation. Totality and invariance are trivial by definition of Z.

For Zig: Suppose $w \xrightarrow{i}_{\mathcal{M}} w'$ and $wZ(w_0, w_1, \ldots, w_n)$ then $w = w_0$. Since $(w, w_1, \ldots, w_n)$ exists in $W^{\mathcal{N}_0 \oplus \cdots \oplus \mathcal{N}_n}$, then $V^{\mathcal{M}}(w) \cap Q^i = V^{\mathcal{N}_i}(w_i)$. Therefore w and w_i satisfy the same set of Boolean formulas based on Q^i. Since $\mathcal{M}$ is locally generated we know that w' and w agree on the formulas in Φ_i. Therefore w_i and w' must also agree on the truth values of the formulas in Φ_i, and thus $w_i \xrightarrow{i}_{\mathcal{M}} w'$. Since $\xrightarrow{j}_{\mathcal{N}_i}$ is universal for $j \neq i$, it is clear that $(w, w_1, \ldots, w_n) \xrightarrow{i} (w', w', \ldots, w')$ in $\mathcal{N}_0 \oplus \cdots \oplus \mathcal{N}_n$ and $w'Z(w', w', \ldots, w')$.

For Zag: Suppose $(w_0, w_1, \ldots, w_n) \xrightarrow{i}_{\mathcal{N}_i} (w'_0, w'_1, \ldots, w'_n)$ and $wZ(w_0, w_1, \ldots, w_n)$, we then have $w = w_0$ and $w_i \xrightarrow{i}_{\mathcal{N}_i} w'_i$. By the definition of $\mathcal{N}_i$, we have $w_i \xrightarrow{i}_{\mathcal{M}} w'_i$. Thus w_i and w'_i agree on formulas in Φ_i. Since w_0 in $\mathcal{M}$ is compatible with w_i in $\mathcal{N}_i$ and w'_i in $\mathcal{N}_i$ is compatible with w'_0 in $\mathcal{M}$, thus w'_0 also agrees with w_0 on formulas in Φ_i. Therefore $w \xrightarrow{i}_{\mathcal{M}} w'_0$ and $w'_0 Z(w'_0, w'_1, \ldots, w'_n)$. This proves the Zag condition.

Now we have shown $\mathcal{M} \leftrightarrows \mathcal{N}_0 \oplus \cdots \oplus \mathcal{N}_n$. Let $\mathcal{M}_i$ be the bisimulation contraction of $\mathcal{N}_i$. From proposition 4, we know $\mathcal{M} \leftrightarrows (\mathcal{M}_0 \oplus \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_n)$. ■

As a corollary, we have:

COROLLARY 23. — *Given a set of agents $I = \{1, 2, \ldots n\}$. Suppose $\mathcal{M} = (W, I, R, V, Q)$ is locally generated w.r.t. $\Phi_1, \ldots, \Phi_n$ and it satisfies the following conditions:*

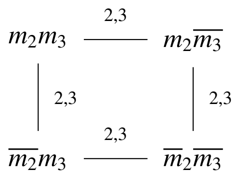
– *for any* $Q' \subseteq Q$ *there is a* $w \in W$ *such that* $V(w) = Q$
– *any propositional letter* $p \in Q$ *appears in some* Φ_i

then there are models $\mathcal{M}_1, \ldots, \mathcal{M}_n$ *such that:*

– $\mathcal{M} \underline{\leftrightarrow} (\mathcal{M}_1 \obar \cdots \obar \mathcal{M}_n)$;
– $|W^{\mathcal{M}_j}| \leq |W|$ *and* $\mathcal{M}_i$ *is bisimulation contracted;*
– $Q^{\mathcal{M}_j} = \{p \in Q^{\mathcal{M}} \mid p \text{ appears in } \Phi_j\}$ *for* $j > 0$*;*

Proof. — From Theorem 22, there are certain $\mathcal{M}_0, \ldots, \mathcal{M}_n$ such that $\mathcal{M} \underline{\leftrightarrow} (\mathcal{M}_0 \obar \mathcal{M}_1 \obar \cdots \obar \mathcal{M}_n)$. Since every possible valuation over Q is witnessed by some state in $\mathcal{M}$ and $\bigcup_{i \in I} Q^{\mathcal{M}_i} = Q = Q^{\mathcal{M}_1 \obar \cdots \obar \mathcal{M}_n}$, every possible state of affairs (valuation) over Q is witnessed by some state in $\mathcal{M}_1 \obar \cdots \obar \mathcal{M}_n$. It is not hard to verify that adding the universal ignorance model $\mathcal{M}_0$ over Q to the composition does not change the model modulo bisimulation and therefore it can be dropped. Indeed, in the proof of the previous theorem, $\mathcal{M}_0$ was used to add extra propositional letters that are not covered by any Φ_i and to rule out unrealisable state of affairs created by the composition. ■

The above corollary gives another way to decompose the Muddy Children models different from the one we gave earlier. Recall that an n-Muddy Children model is locally generated by sets of observables $\Phi_1, \ldots, \Phi_n$ where $\Phi_i = \{m_j \mid j \neq i, j \in I\}$. For example, if $n = 3$ then the set of observables for agent 1 is $\{m_2, m_3\}$. We can then decompose the 3-Muddy Children model $\mathcal{M}$ by $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$, where e.g., $\mathcal{M}_1$ is as follows:



Compared to the two-world model decomposition in Section 2, the above decomposition requires bigger size components (2^{n-1} worlds for the n children case). This is because we decompose the model in an *agent-based* fashion: each component represents one agent's observational power regardless of the others. Thus if the vocabulary of the set of observables Φ_i is big then so is the component model. In the Muddy Children example, if there are more children then the vocabulary of the observables for each child also increases (e.g., new m_j), therefore the component model for this agent also grows bigger. However, in other applications the vocabulary of observables may not increase even when the initial model grows bigger.

To decompose a Muddy Children model as in Section 2, we decompose the model in an *issue-based* fashion (every proposition is an issue), as the following theorem demonstrates:

Theorem 24 (Decomposition by issues). — *Given a set of agents* $I = \{1, 2, \ldots n\}$ *and a set of propositional letters* $Q = \{p_1, \ldots, p_k\}$*, if* $\mathcal{M} = (W, I, R, V, Q)$ *is locally generated by* $\Phi_1, \ldots, \Phi_n$ *such that* Φ_i *only contains atomic propositions (i.e.,* $\Phi_i \subseteq Q$*), then there are models* $\mathcal{M}_1, \ldots, \mathcal{M}_k$ *and* $\mathcal{M}_0$ *such that:*

– $\mathcal{M} \underline{\leftrightarrow} (\mathcal{M}_0 \obar \mathcal{M}_1 \obar \cdots \obar \mathcal{M}_k)$;

– $Q^{\mathcal{M}_j} = \{p_j\}$ *for* $j > 0$ *and* $Q^0 = Q$;
– $|W^{\mathcal{M}_j}| = 2$ *for* $j > 0$.

PROOF. — Let $\mathcal{M}_0$ be the same as in the proof of Theorem 22. For $j > 0$, let $\mathcal{M}_j = (W_j, I, R^{\mathcal{M}_j}, V_j, Q^j)$ where:

– $W_j = \{p_j, \overline{p_j}\}$;
– $Q^j = \{p_j\}$;
– $V_j(p_j) = \{p_j\}$ and $V_j(\overline{p_j}) = \emptyset$ with the obvious interpretation;
– for $j > 0 : w \xrightarrow{i}_{\mathcal{M}_j} v \iff w = v$ or ($w \neq v$ and $p_j \notin \Phi_i$).

Let a relation $Z \subseteq W^{\mathcal{M}} \times W^{\mathcal{M}_0 \oplus \cdots \oplus \mathcal{M}_k}$ be given as follows:

$$wZ(w_0, w_1, \ldots, w_k) \iff w = w_0$$

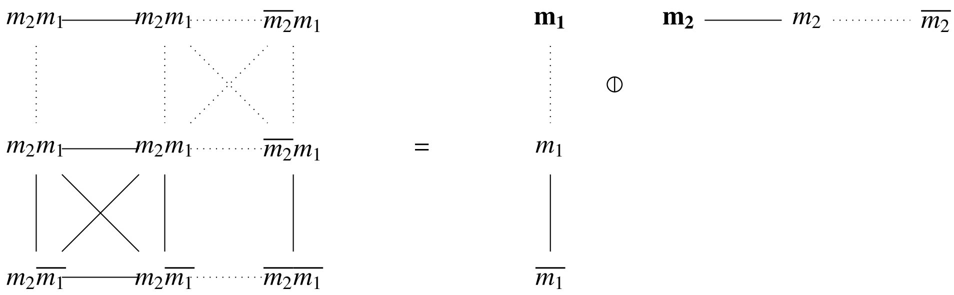
If $wZ(w_0, w_1, \ldots, w_k)$ then for $0 < j \leq k : w_j$ intuitively represents the truth value of p_j in w. Now let us verify that Z is indeed a total bisimulation. Totality and invariance are trivial.

For Zig: Suppose $w \xrightarrow{i}_{\mathcal{M}} w'$ and $wZ(w_0, w_1, \ldots, w_k)$. Note that $(w_0, w_1, \ldots, w_k)$ exists in $W^{\mathcal{M}_0 \oplus \cdots \oplus \mathcal{M}_k}$. It follows that $w_0 = w$ and ($w_j = p_j \iff p_j \in V^{\mathcal{M}}(w)$). Let $w'_j = p_j$ if $p_j \in V^{\mathcal{M}}(w')$, and $w'_j = \overline{p_j}$ otherwise. It is not hard to see that $(w', w'_1, \ldots, w'_k)$ in $\mathcal{M}_0 \oplus \cdots \oplus \mathcal{M}_k$ such that $w'Z(w', w'_1, \ldots, w'_k)$. We need to show that for each $j > 0 : w_j \xrightarrow{i}_{\mathcal{M}_j} w'_j$. If $w_j = w'_j$ then $w_j \xrightarrow{i}_{\mathcal{M}_j} w'_j$ by definition of $\xrightarrow{i}_{\mathcal{M}_j}$. Now suppose $w_j \neq w'_j$. Since $\mathcal{M}$ is locally generated by $\Phi_1, \ldots, \Phi_n$, w and w' agree on the truth values of the atomic propositions in Φ_i. Therefore $p_j \notin \Phi_i$, thus $w_j \xrightarrow{i}_{\mathcal{M}_j} w'_j$.

For Zag: Suppose $(w_0, w_1, \ldots, w_k) \xrightarrow{i}_{\mathcal{M}_i} (w'_0, w'_1, \ldots, w'_k)$ and $wZ(w_0, w_1, \ldots, w_k)$, we have $w_0 = w$ and for any $j \leq k : w_j \xrightarrow{i}_{\mathcal{M}_j} w'_j$. By the definition of $\mathcal{M}_i$, we have $w_j = w'_j$, or $w_j \neq w'_j$ and $p_j \notin \Phi_i$. Namely, for $p_j \in \Phi_i : w_j = w'_j$. Therefore w and w' agree on the truth values of the propositions in Φ_i. Since $\mathcal{M}$ is locally generated by $\Phi_1, \ldots, \Phi_n$ we get $w \xrightarrow{i}_{\mathcal{M}} w'$. This proves the Zag condition. ■

Similar to the previous corollary, we can drop the universal ignorance model $\mathcal{M}_0$ in the decomposition under the condition that $\mathcal{M}$ witnessed all the possible state of affairs. According to the above theorem, a locally generated model by sets of atomic propositions can be decomposed by components based on each atomic proposition. This gives us the desired decomposition of the n-Muddy Children models as in Section 2.

Theorems 22 and 24 show that we can decompose a locally generated model. On the other hand, there are models which are not locally generated but decomposable in a non-trivial way. For example, consider the model on the left below (to ease the presentation, we use solid lines for agent 1 and dotted lines for agent 2):



This model is not bisimilar to any propositionally differentiated model. From proposition 20 it follows that it is not bisimilar to any locally-generated model. Nevertheless, $\mathcal{M}$ can be decomposed into two models as the above composition on the right shows.

If we take the boldface states as the real worlds in these two component models respectively, then the two models capture the situations where agent 2 is not sure whether 1 knows m_1 and agent 1 is not sure whether 2 knows m_2. If we interpret m_1 and m_2 as in Muddy Children, then the composed model, when taking the top-left corner state as the real world, captures the situation where the children can see each other's faces but are not sure whether the other has a mirror (actually they do have mirrors). Since the vocabularies of the above two models are disjoint, from proposition 16, we know that any true claim about only m_2 or m_1 will be preserved at the components. For example, agent 1 knows agent 2 does not know whether agent 1 knows m_1 can be verified in the left hand component model.

5. Connections and future Work

In (Dunne *et al.*, 2008) 'cooperative boolean games' are studied: games where agents cooperatively can achieve goals stated as propositional formulas. In the present framework, the variables under the control of an agent can be taken to be the variables that are in the domain of a substitution in component models representing the perceptive and control abilities of agents. This points the way towards extending cooperative boolean games with an epistemic dimension, and for building a logical framework for the study of cooperative epistemic games.

If we interpret a partial vocabulary as the set of propositions that agents are aware of, then our vocabulary expansion can be viewed as a way of modelling certain *dynamics of awareness*, which is studied in (van Benthem *et al.*, 2009). The vocabulary expansion in our framework is a global action expanding the awareness of all agents simultaneously, while (van Benthem *et al.*, 2009) studies 'awareness raising' actions of individual agents. As for the connection with the awareness logics studied in (Heifetz *et al.*, 2008) and (Halpern *et al.*, 2009): we may view their multi-level awareness models as a collection of models generated from an ignorance model by performing expansion operations. A more detailed comparison is left for future work.

Our approach of composing epistemic models from small components differs from the symmetry reduction technique of (Cohen *et al.*, 2009) where the key tool is agent permutation, while in our approach the idea is to merge models with partial vocabularies. As we have demonstrated, small components may carry precise information about the composite model which may shed light on multi-agent model checking (cf. (Wang, 2010) for more details on this particular aspect). Therefore we intend to extend the epistemic model checker DEMO (van Eijck, 2007) with model composition operations, to investigate the practical usefulness of the approach.

There is no reason to restrict the proposal of the paper for studying extensions of vocabulary to the propositional part of the language. The same can be done for the relational part, by extending a model with new agents. The extension could start with representing all possibilities for what the new agent might know, but more specific options are also worth exploring, such as an initial belief that a new agent knows nothing (the way the Greeks thought of the Barbarians).

There are also more general questions about decomposition: for which (finite) $\mathcal{M}$ is it possible to find a decomposition of $\mathcal{M}$ (modulo bisimulation) such that each component has strictly smaller vocabulary than $\mathcal{M}$? Or more general still: Is there a *normal form* of $\mathcal{M}$ by composition and relativization (public announcement)? Results on factor products from graph theory may help here.

Finally, the combination of communicative actions and vocabulary expansion deserves separate study. A first task here could be to axiomatize the strong Kleene logic (cf. (Kleene, 1950)) of public announcement $!\phi$ and vocabulary expansion $\sharp p$, where $\sharp p$ is interpreted as the model changing operation $\mathcal{M} \mapsto \mathcal{M} \triangleleft \{p\}$.

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Appendix

A. Proof of Lemma 17

Proof. — To ease the presentation, we first introduce some notations. For a Boolean precondition $T^{\mathcal{A}}(e)$ of e, a vocabulary Q such that $Q^{\mathcal{A}} \subseteq Q$, and a set $X \subseteq Q$, we write $X \vDash_Q T^{\mathcal{A}}(e)$ if X (viewed as a valuation for Q) makes $T^{\mathcal{A}}(e)$ true. It is clear that $X \cap Q^{\mathcal{A}} \vDash_Q T^{\mathcal{A}}(e)$ iff $X \vDash_Q T^{\mathcal{A}}(e)$. In case $Q = \bigcup_{j\in J} Q^{\mathcal{M}_j}$ we write $X \vDash_{\{\mathcal{M}_j | j\in J\}} T^{\mathcal{A}}(e)$.

We need to show for any state u that exists in $\mathcal{M}_1 = (\mathcal{M} \oplus \mathcal{N}) \circledcirc \mathcal{A}$ there is a state v in $\mathcal{M}_2 = (\mathcal{M} \circledcirc \mathcal{A}) \oplus (\mathcal{N} \circledcirc \mathcal{A})$ such that uZv, and for any v in $\mathcal{M}_2$ there is a u in $\mathcal{M}_1$ such that uZv. Suppose $((s,t,X),e)$ exists in $\mathcal{M}_1$. Then then the following hold:

Fact 1 $X \subseteq Q^{\mathcal{A}} - (Q^{\mathcal{M}} \cup Q^{\mathcal{N}})$;

Fact 2 $V^{\mathcal{M}}(s) \cap Q^{\mathcal{N}} = V^{\mathcal{N}}(t) \cap Q^{\mathcal{M}}$;

Fact 3 $V^{\mathcal{M}}(s) \cup V^{\mathcal{N}}(t) \cup X \vDash_{\mathcal{M},\mathcal{N},\mathcal{A}} T^{\mathcal{A}}(e)$.

Now we let:

$$X_1 = X \cup ((V^{\mathcal{N}}(t) - V^{\mathcal{M}}(s)) \cap Q^{\mathcal{A}}) \text{ and } X_2 = X \cup ((V^{\mathcal{M}}(s) - V^{\mathcal{N}}(t)) \cap Q^{\mathcal{A}}).$$

Clearly $X = X_1 \cap X_2$. To show $((s,X_1,e),(t,X_2,e))$ exists in $\mathcal{M}_2$, we need to show:

1) X_1 and X_2 are well-defined: $X_1 \subseteq Q^{\mathcal{A}} - Q^{\mathcal{M}}$ and $X_2 \subseteq Q^{\mathcal{A}} - Q^{\mathcal{N}}$.

2) e can be executed on both (s,X_1) and (t,X_2): $V^{\mathcal{M}}(s) \cup X_1 \vDash_{\mathcal{M},\mathcal{A}} T^{\mathcal{A}}(e)$ and $V^{\mathcal{N}}(t) \cup X_2 \vDash_{\mathcal{N},\mathcal{A}} T^{\mathcal{A}}(e)$.

3) (s,X_1,e) and (t,X_2,e) can be composed: $(V^{\mathcal{M}}(s) \cup X_1) \cap (Q^{\mathcal{N}} \cup Q^{\mathcal{A}}) = (V^{\mathcal{N}}(t) \cup X_2) \cap (Q^{\mathcal{M}} \cup Q^{\mathcal{A}})$.

For (1): Recall that $V^{\mathcal{M}}(s) \cap Q^{\mathcal{N}} = V^{\mathcal{N}}(t) \cap Q^{\mathcal{M}}$, thus we have $V^{\mathcal{N}}(t) \cap Q^{\mathcal{M}} \subseteq V^{\mathcal{M}}(s)$ and $V^{\mathcal{M}}(s) \cap Q^{\mathcal{N}} \subseteq V^{\mathcal{N}}(t)$. Therefore, $(V^{\mathcal{N}}(t) - V^{\mathcal{M}}(s)) \cap Q^{\mathcal{M}} = \emptyset$ and $(V^{\mathcal{M}}(s) - V^{\mathcal{N}}(t)) \cap Q^{\mathcal{N}} = \emptyset$. It means $((V^{\mathcal{N}}(t) - V^{\mathcal{M}}(s)) \cap Q^{\mathcal{A}}) \subseteq Q^{\mathcal{A}} - Q^{\mathcal{M}}$ and $((V^{\mathcal{M}}(s) - V^{\mathcal{N}}(t)) \cap Q^{\mathcal{A}}) \subseteq Q^{\mathcal{A}} - Q^{\mathcal{N}}$. Also note that $X \subseteq Q^{\mathcal{A}} - (Q^{\mathcal{M}} \cup Q^{\mathcal{N}}) \subseteq Q^{\mathcal{A}} - Q^{\mathcal{M}}$ and similarly $X \subseteq Q^{\mathcal{A}} - Q^{\mathcal{N}}$. Therefore by the definitions of X_1 and X_2 we have $X_1 \subseteq Q^{\mathcal{A}} - Q^{\mathcal{M}}$ and $X_2 \subseteq Q^{\mathcal{A}} - Q^{\mathcal{N}}$.

For (2): By the definition of X_1:

$$V^{\mathcal{M}}(s) \cup X_1 = V^{\mathcal{M}}(s) \cup X \cup ((V^{\mathcal{N}}(t) - V^{\mathcal{M}}(s)) \cap Q^{\mathcal{A}})$$

Then we have:

$$
\begin{aligned}
&(V^{\mathcal{M}}(s) \cup X_1) \cap Q^{\mathcal{A}} \qquad (\#)\\
&= (V^{\mathcal{M}}(s) \cup X \cup ((V^{\mathcal{N}}(t) - V^{\mathcal{M}}(s)) \cap Q^{\mathcal{A}})) \cap Q^{\mathcal{A}}\\
&= ((V^{\mathcal{M}}(s) \cup X) \cap Q^{\mathcal{A}}) \cup ((V^{\mathcal{N}}(t) - V^{\mathcal{M}}(s)) \cap Q^{\mathcal{A}})\\
&= (V^{\mathcal{M}}(s) \cup V^{\mathcal{N}}(t) \cup X) \cap Q^{\mathcal{A}}
\end{aligned}
$$

Since $V^{\mathcal{M}}(s) \cup V^{\mathcal{N}}(t) \cup X \vDash_{\mathcal{M},\mathcal{N},\mathcal{A}} T^{\mathcal{A}}(e)$ we have

$$(V^{\mathcal{M}}(s) \cup V^{\mathcal{N}}(t) \cup X) \cap Q^{\mathcal{A}} \vDash_{\mathcal{M},\mathcal{N},\mathcal{A}} T^{\mathcal{A}}(e)$$

Therefore from the derivation (#), $(V^{\mathcal{M}}(s) \cup X_1) \cap Q^{\mathcal{A}} \vDash_{\mathcal{M},\mathcal{N},\mathcal{A}} T^{\mathcal{A}}(e)$ and then $V^{\mathcal{M}}(s) \cup X_1 \vDash_{\mathcal{M},\mathcal{A}} T^{\mathcal{A}}(e)$. Similarly we can prove $V^{\mathcal{N}}(t) \cup X_2 \vDash_{\mathcal{N},\mathcal{A}} T^{\mathcal{A}}(e)$.

For (3): By the definition of X_1:

$$(V^{\mathcal{M}}(s) \cup X_1) \cap (Q^{\mathcal{N}} \cup Q^{\mathcal{A}}) = ((V^{\mathcal{M}}(s) \cup X_1) \cap Q^{\mathcal{A}}) \cup ((V^{\mathcal{M}}(s) \cup X_1) \cap Q^{\mathcal{N}})$$

From (#), we know that: $(V^{\mathcal{M}}(s) \cup X_1) \cap Q^{\mathcal{A}} = (V^{\mathcal{M}}(s) \cup V^{\mathcal{N}}(t) \cup X) \cap Q^{\mathcal{A}}$, thus

$$(V^{\mathcal{M}}(s) \cup X_1) \cap (Q^{\mathcal{N}} \cup Q^{\mathcal{A}}) = ((V^{\mathcal{M}}(s) \cup V^{\mathcal{N}}(t) \cup X) \cap Q^{\mathcal{A}}) \cup (V^{\mathcal{M}}(s) \cap Q^{\mathcal{N}}) \cup (X_1 \cap Q^{\mathcal{N}}) \quad (\dagger)$$

Note that

$$
\begin{aligned}
&X_1 \cap Q^{\mathcal{N}}\\
&= (X \cup ((V^{\mathcal{N}}(t) - V^{\mathcal{M}}(s)) \cap Q^{\mathcal{A}})) \cap Q^{\mathcal{N}}\\
&= ((V^{\mathcal{N}}(t) - V^{\mathcal{M}}(s)) \cap Q^{\mathcal{A}}) \cap Q^{\mathcal{N}} \qquad (\text{ since } X \cap Q^{\mathcal{N}} = \emptyset)\\
&= (V^{\mathcal{N}}(t) - V^{\mathcal{M}}(s)) \cap Q^{\mathcal{A}} \qquad (\text{ since } V^{\mathcal{N}}(t) \subseteq Q^{\mathcal{N}})
\end{aligned}
$$

Therefore going back to (†) we have:

$$
\begin{aligned}
&(V^{\mathcal{M}}(s) \cup X_1) \cap (Q^{\mathcal{N}} \cup Q^{\mathcal{A}})\\
&= ((V^{\mathcal{M}}(s) \cup V^{\mathcal{N}}(t) \cup X) \cap Q^{\mathcal{A}}) \cup (V^{\mathcal{M}}(s) \cap Q^{\mathcal{N}}) \cup ((V^{\mathcal{N}}(t) - V^{\mathcal{M}}(s)) \cap Q^{\mathcal{A}})\\
&= ((V^{\mathcal{M}}(s) \cup V^{\mathcal{N}}(t) \cup X) \cap Q^{\mathcal{A}}) \cup (V^{\mathcal{M}}(s) \cap Q^{\mathcal{N}}) \qquad (\ddagger)
\end{aligned}
$$

Similarly we can show

$$(V^{\mathcal{N}}(t) \cup X_2) \cap (Q^{\mathcal{M}} \cup Q^{\mathcal{A}}) = ((V^{\mathcal{M}}(s) \cup V^{\mathcal{N}}(t) \cup X) \cap Q^{\mathcal{A}}) \cup (V^{\mathcal{N}}(t) \cap Q^{\mathcal{M}}) \qquad (\S)$$

From the **Fact 1** ($V^{\mathcal{N}}(t) \cap Q^{\mathcal{M}} = V^{\mathcal{M}}(s) \cap Q^{\mathcal{N}}$), (‡) and (§) we have:

$$(V^{\mathcal{M}}(s) \cup X_1) \cap (Q^{\mathcal{N}} \cup Q^{\mathcal{A}}) = (V^{\mathcal{N}}(t) \cup X_2) \cap (Q^{\mathcal{M}} \cup Q^{\mathcal{A}})$$

This proves (3).

Till now we have proved that for any state u that exists in $\mathcal{M}_1$ there is a state v exists in $\mathcal{M}_2$ such that uZv. Now suppose $((s, X_1, e), (t, X_2, e'))$ exists in $\mathcal{M}_2$ we need to show that there is a u in $\mathcal{M}_1$ such that uZv. Since $\mathcal{A}$ is propositionally differentiated, no two actions can be executed under the same valuation over $Q^{\mathcal{A}}$, thus events $e = e'$. We now only need to show that $((s, t, X_1 \cap X_2), e)$ exists in $\mathcal{M}_1$. Formally we need to verify the following claims:

1) s and t are compatible.

2) X is well-defined: $X_1 \cap X_2 \subseteq Q^{\mathcal{A}} - (Q^{\mathcal{M}} \cup Q^{\mathcal{N}})$.
3) e can be executed on $(s, t, X_1 \cap X_2)$: $V^{\mathcal{M}}(s) \cup V^{\mathcal{N}}(t) \cup (X_1 \cap X_2) \vDash_{\mathcal{M},\mathcal{N},\mathcal{A}} T^{\mathcal{A}}(e)$.

From the existence of $((s, X_1, e), (t, X_2, e))$, clearly s and t can be composed. Since $X_1 \subseteq Q^{\mathcal{A}} - Q^{\mathcal{M}}$ and $X_2 \subseteq Q^{\mathcal{A}} - Q^{\mathcal{N}}$, (2) is also straightforward. Now we prove (3). Since (s, X_1, e) and (t, X_2, e') can be composed

$$V^{\mathcal{M}}(s) \cup X_1 \cup V^{\mathcal{N}}(t) = V^{\mathcal{M}}(s) \cup X_2 \cup V^{\mathcal{N}}(t)$$

Now we have:

$$\begin{aligned}
&V^{\mathcal{M}}(s) \cup X_1 \cup V^{\mathcal{N}}(t) \\
&= (V^{\mathcal{M}}(s) \cup X_1 \cup V^{\mathcal{N}}(t)) \cap (V^{\mathcal{M}}(s) \cup X_2 \cup V^{\mathcal{N}}(t)) = (V^{\mathcal{M}}(s) \cup V^{\mathcal{N}}(t) \cup (X_1 \cap X_2))
\end{aligned}$$

Since $((s, X_1, e), (t, X_2, e))$ exists, it is not hard to see that $(V^{\mathcal{M}}(s) \cup V^{\mathcal{N}}(t) \cup X_1) \cap Q^{\mathcal{A}} \vDash_{\mathcal{M},\mathcal{N},\mathcal{A}} T^{\mathcal{A}}(e)$. Thus $V^{\mathcal{M}}(s) \cup V^{\mathcal{N}}(t) \cup (X_1 \cap X_2) \vDash_{\mathcal{M},\mathcal{N},\mathcal{A}} T^{\mathcal{A}}(e)$. ■