Representing Imperfect Information of Procedures with Hyper Models

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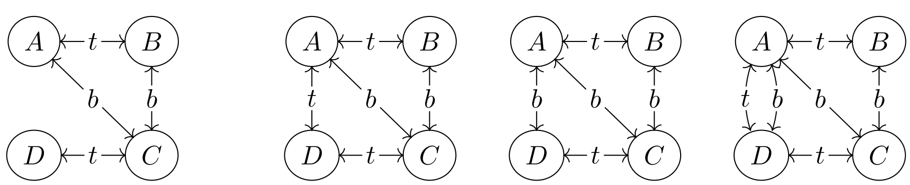
Abstract. When reasoning about knowledge of procedures under imperfect information, the explicit representation of epistemic possibilities blows up the S5-like models of standard epistemic logic. To overcome this drawback, in this paper, we propose a new logical framework based on compact models without epistemic accessibility relations for reasoning about knowledge of procedures. Inspired by the 3-valued abstraction method in model checking, we introduce hyper models which encode the imperfect procedural information. We give a highly non-trivial 2-valued semantics of epistemic dynamic logic on such models while validating all the usual S5 axioms. Our approach is suitable for applications where procedural information is 'learned' incrementally, as demonstrated by various examples.

1 Introduction

Suppose there are four cities A, B, C, D which are connected by public transportation as the following leftmost map shows ($\xrightarrow{b}$ for bus and $\xrightarrow{t}$ for train):



We may view the map as a Kripke model and use various modal logics such as Propositional Dynamic Logic (PDL) [12] to describe routes or more complicated trip plans from one city to others. Now suppose we are informed that A and D will also be connected next year, but it is not clear whether it will be a bus line or a train connection or even both. Then the new map can be any one of the three right-hand-side maps above. Although the new information is *imperfect*, we still can *know* that city D will become directly reachable from A since this is true in all the possible new maps, and it may be *possible* to reach C from A by train via D, since this is true in some possible maps.

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As we have seen, imperfect information about the connectivity of the states introduces uncertainty. To encode such uncertainty, two-dimensional Kripke models are often used, which not only have labelled transitions but also epistemic accessibility relations between states, such as the epistemic temporal models in [10,4,11,15] and the models of imperfect information games in game theory (cf. e.g., [9]). In such epistemic logical frameworks, a proposition is known on a state if it is true on all the possible states linked with the current state by the epistemic relation.

However, explicitly representing possibilities as in the usual epistemic logical frameworks leads to the problem of state space explosion even in very simple case, if there is considerable ignorance. For example, if we have no idea how those four cities are connected by possibly four transportation methods (e.g., train, bus, flight, boat), then there are at least $(2^{4*3})^4 = 2^{48} > 10^{14}$ possible maps, which is in the order of the number of cells in a human body or the number of neuronal connections in a human brain. It is clear that in reality we do not go through all these possibilities in our mind to acquire knowledge. This leads to our first research question:

> *Can we have a compact alternative model of epistemic logic of procedures without explicitly representing all the possibilities?*

Moreover, it is important to incorporate new (imperfect) information about procedures into the current model which allows us to incrementally build up the model even from scratch. The new information may be given in a syntactic form which uses implicit quantifiers, such as '*there is* a bus going from A to *either* B or C, but I am not quite sure which due to the recent change of routes. On the other hand, from both B or C you should be able to reach D by *some* public transportation.' The imperfect information may also be given by a complicated procedure which is not just one-step, e.g., 'taking a bus then a train will get you home'. However, the semantic way of incorporating new information is usually done by essentially *eliminating* inconsistent possibilities according to the new information in epistemic frameworks such as dynamic epistemic logic [2], which again assumes that all the possibilities are represented in the model. Thus the following question is another challenge:

> *How to incorporate new procedural information semantically in an incremental fashion?*

The technical contribution of this paper is a new semantics-driven epistemic logical framework aiming at solving the above questions. Note that we may alternatively represent all the imperfect information syntactically but in the case of procedural information, the graphic models are more natural and compact for very complicated procedural information between states. Moreover, doing model checking is in general computationally more efficient than theorem proving. In the rest of this introduction, we will explain our main ideas.

Main ideas

To handle the first question, we need a way to encode possibilities in a compact and implicit way. Our approach is inspired by the 3-valued abstraction technique in model checking (see [5] for an overview). To handle the problem of state space explosion in model checking, the abstraction technique makes a Kripke model smaller by abstracting away some information. Clearly, the smaller abstract model may not preserve the truth value of all the formulas in concern, but a suitable 3-valued semantics on abstractions can make sure the following:

- formulas that are true in the abstraction are also true in the original model
- formulas that are false in the abstraction are also false in the original model

In our point of view, an abstraction can be indeed viewed as a compact representation of potential concrete models which are consistent with the information represented in the abstract model. The information of transitions in the abstract model is typically encoded by two special kinds of transitions which are under- and over-approximations of the ones in the actual model [13]. This inspired us to use similar abstract transitions to encode imperfect procedural information. The final source of inspiration is from [3,1] where the approximations of the transitions are labelled by regular expressions and this helps us in dealing with arbitrarily complicated procedural information expressed by regular expressions.

Based on these ideas, we propose *hyper models* to encode the imperfect procedural information, and an epistemic PDL defined on such models, with the following features:

- Hyper models assemble possibilities in an implicit and compact way using abstract transitions (labelled by regular expressions) but not epistemic accessibility relations.
- They incorporate new information incrementally by *adding* new transitions.
- The semantics of our language is defined on the hyper models *directly*, and there is no need to unpack the hyper models into numerous possible Kripke models.
- The logic is still 2-valued and *all* the usual **S5** axioms are valid, but *not* the necessitation rule which may cause logical omniscience.

Of course, there is also a price to pay: not all the collections of Kripke models are representable by hyper models, to which we will come back in Section 4. In the rest of the paper, we first formalize imperfect procedural information in Section 2, and then introduce and discuss simple and full hyper models in Section 3. Finally we show that hyper models are indeed compact representations of Kripke models and point out future directions in Section 4.

2 Preliminaries

Kripke models are used to represent how states are connected by atomic actions:

Definition 1 (Kripke model). *Given a set of basic propositional letters* $\mathbf{P}$*, and a set of atomic action symbols* $\mathbf{\Sigma}$*, a Kripke model* $\mathcal{M}$ *over* $\mathbf{P}$ *and* $\mathbf{\Sigma}$ *is a tuple* $(S, \rightarrow, V)$ *where:*

- S *is a non-empty set of states.*
- $\rightarrow \subseteq S \times \mathbf{\Sigma} \times S$ *is a binary relation over* S *labelled by action symbols from* $\mathbf{\Sigma}$*.*
- $V\colon S \rightarrow 2^{\mathbf{P}}$ *assigns to each state a set of basic propositional letters.*

We write $s \xrightarrow{a} t$ *if* $(s, a, t) \in \rightarrow$*. Given* $w = a_1 a_2 \ldots a_n$ *we write* $s \xrightarrow{w} t$ *if there are* $s_0, s_1, \ldots, s_n$ *such that* $s_0 = s$ *and* $s_n = t$ *and* $s_k \xrightarrow{a_k} s_{k+1}$ *for all* $0 \leq k < n$*.*

Note that $\mathcal{M}$ may not be deterministic in the sense that for some $s \in S_{\mathcal{M}}$ and some $a \in \mathbf{\Sigma}$ there may be *more than* one t such that $s \xrightarrow{a} t$. Intuitively this means that doing a on state s may result in different states due to some external factors which are not modelled in $\mathcal{M}$.

The simplest procedure is a one-step atomic action $a \in \mathbf{\Sigma}$, based on which more complicated procedures are constructed as regular expressions:

$$\pi \quad ::= \quad a \mid \pi;\pi \mid \pi + \pi \mid \pi^*$$

where $a \in \mathbf{\Sigma}$. Intuitively, ; is the sequential composition, $+$ is the non-deterministic choice, and * is the iteration operation. Let $\mathit{\Pi}_{\mathbf{\Sigma}}$ denote the set of all regular expressions based on $\mathbf{\Sigma}$. The set of action sequences denoted by a regular expression π (notation: $\mathcal{L}(\pi)$) is defined as follows:

$$\begin{array}{l}
\mathcal{L}(a) = \{a\} \\
\mathcal{L}(\pi;\pi') = \{wv \mid w \in \mathcal{L}(\pi) \text{ and } v \in \mathcal{L}(\pi')\} \\
\mathcal{L}(\pi + \pi') = \mathcal{L}(\pi) \cup \mathcal{L}(\pi') \\
\mathcal{L}(\pi^*) = \{\epsilon\} \cup \bigcup_{n>0}(\mathcal{L}(\underbrace{\pi;\cdots;\pi}_{n}))
\end{array}$$

where ϵ is the empty sequence. In the sequel, we abuse the notation by writing $w \in \pi$ for $w \in \mathcal{L}(\pi)$ and writing $\pi \subseteq \pi'$ for $\mathcal{L}(\pi) \subseteq \mathcal{L}(\pi')$.

We use the language of PDL to describe the procedures encoded in a Kripke model:

$$\phi \quad ::= \quad \top \mid p \mid \neg\phi \mid \phi \wedge \phi \mid \langle\pi\rangle\phi$$

where $\langle\pi\rangle\phi$ formulas are interpreted on pointed Kripke model $\mathcal{M}, s$ as follows (cf. e.g., [7]):

$$\boxed{\mathcal{M}, s \Vdash \langle\pi\rangle\phi \iff \text{there exists a } t \text{ such that } s \xrightarrow{w} t \text{ for some } w \in \pi \text{ and } t \Vdash \phi}$$

We say $s_0, s_1, \ldots, s_n$ is an *execution* of π if there exist $w = a_1, a_2, \ldots, a_n \in \pi$ such that $s_k \xrightarrow{a_{k+1}} s_{k+1}$ for $0 \leq k < n$. In another word, $\langle\pi\rangle\phi$ is true at s iff there is an execution of π from s to a ϕ*-state*, i.e., a state where ϕ holds.

Now, we formalize a piece of imperfect procedure information as a Hoare-like triple $\langle\phi, X, \psi\rangle$ where ϕ and ψ are PDL formulas and X is one of $\pi^{\exists}$ or $\pi^{\forall}$ where $\pi \in \mathit{\Pi}_{\mathbf{\Sigma}}$. ϕ and ψ denote the *precondition* (initial states) and the *postcondition* (goal states) of the procedure respectively, and π denotes the procedure quantified by $\exists$ or $\forall$. Intuitively, the quantifiers work as follows:

- $\langle \phi,\ \pi^{\exists},\ \psi \rangle$ says that if ϕ holds then there *exists* an execution of π which can make ψ true, e.g., "One of these two buses will get you to the university from home."
- $\langle \phi,\ \pi^{\forall},\ \psi \rangle$ says that if ϕ holds then *all* the executions of π will make sure ψ, e.g., "All the buses departing here will get you to the university."

The *correctness* of a piece of information $\langle \phi, X, \psi \rangle$ is define below, given a Kripke model $\mathcal{M}$:

- $\langle \phi,\ \pi^{\exists},\ \psi \rangle$ is correct iff ($\forall t \Vdash \phi$, $\exists w \in \pi \exists t' : t \xrightarrow{w} t'$ and $t' \Vdash \psi$) iff $\mathcal{M} \Vdash \phi \to \langle \pi \rangle \psi$
- $\langle \phi,\ \pi^{\forall},\ \psi \rangle$ is correct iff ($\forall t \Vdash \phi$, $\forall w \in \pi \forall t'$: if $t \xrightarrow{w} t'$ then $t' \Vdash \psi$) iff $\mathcal{M} \Vdash \phi \to [\pi]\psi$

3 Hyper models

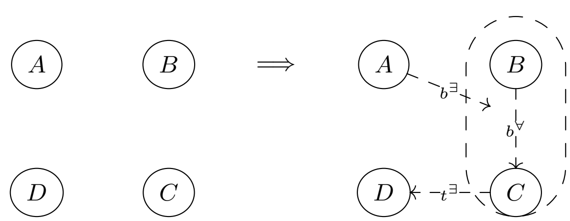
Here is an example to motivate our definition of hyper models:

You have no idea how four cities A, B, C, D are connected (by train or bus). Now suppose that someone tells you that there is a bus going to either B or C from A, there is a train connection from C to D, and all the buses departing from B are going to C. Now what do you know about the route from A to D?

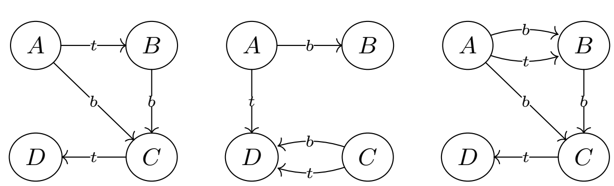
Again, let b denote the bus connections and let t denote the train connections. Let p_x be the basic proposition denoting the location of town x for $x \in \{A, B, C, D\}$. The imperfect procedural information can be formalized as:

$$\langle p_A,\ b^{\exists},\ p_B \vee p_C \rangle, \quad \langle p_C,\ t^{\exists},\ p_D \rangle, \quad \langle p_B,\ b^{\forall},\ p_C \rangle.$$

The simple-minded learning process is to add those information as special transitions in the map, as illustrated below (note that the $b^{\exists}$ transition is from A to $\{B, C\}$):



Given that the information is truthful, the real situation is still not yet determined, for example, the following are three of the possible actual situations consistent with the information available:



However, the agent should *know* the following, which may help him to go to D from A:

There is a bus from A to either B or C, and if it reaches C then D can be reached by a train, otherwise take any bus (if available) from B to get C first in order to reach D finally.

In the rest of this section, we will introduce hyper models formally, and a semantics for *epistemic PDL* (*EPDL*) based on them to reason about knowledge of procedures.

3.1 Models with simple procedural information

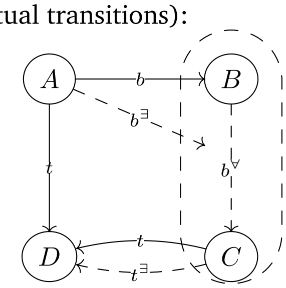
This subsection is a technical warm-up for the next one. We only consider simple procedures ($a^\forall$ or $a^\exists$) based on singleton sets of initial states. To represent such information, we introduce the *simple hyper models* based on Kripke models with extra transitions labelled by $a^\forall$ or $a^\exists$ from a *single* state to a *set* of states:

Definition 2 (Simple hyper model). *A simple hyper model is a tuple $(S, \to, \to_\exists, \to_\forall, V)$ where:*

- *$(S, \to, V)$ is a Kripke model w.r.t. $\mathbf{\Sigma}$.*
- *$\to_\exists \subseteq S \times \mathbf{\Sigma} \times 2^S$ is a labelled binary relation from a state to a set of states.*
- *$\to_\forall \subseteq S \times \mathbf{\Sigma} \times 2^S$ is a labelled binary relation from a state to a set of states.*
- *for all $s \in S, T \subseteq S$: $s \xrightarrow{a}_\exists T$ implies that there exists $t \in T$ such that $s \xrightarrow{a} t$.*
- *for all $s \in S, T \subseteq S$: $s \xrightarrow{a}_\forall T$ implies that for all $t \in S$: $s \xrightarrow{a} t$ implies $t \in T$.*

In (simple) hyper models, $\to$ represents the *actual* transitions between the states, and $\to_\exists$ and $\to_\forall$ represent the available imperfect procedural information to an agent. The last two conditions are crucial to guarantee the correctness of procedural information in the model. Note that this model is from the modeller's point of view, and the agent's knowledge only depends on $\to_\forall$ and $\to_\exists$ but not $\to$, as it will become clear in the semantics of the logic. Here we include the actual transitions in order to validate whether our logic, to be defined later, is a proper epistemic logic, e.g., whether everything the agent knows is actually true. When representing the agent's procedural information only, we can simply leave out the actual transitions given $\to_\forall$ and $\to_\exists$ are reliable. Note that $s \xrightarrow{a}_\forall \emptyset$ denotes 'negative' information: there is no a-transition from s. On the other hand, it is impossible to have $s \xrightarrow{a}_\exists \emptyset$ due to the first correctness condition.

Note that, the transitions $\to_\exists$ and $\to_\forall$ are *not* defined by $\to$. As an example, recall the model we mentioned at the beginning of this section (now with the actual transitions):



where:

- $S = \{A, B, C, D\}$,
- $\to = \{(A, b, B), (A, t, D), (C, t, D)\}$,
- $\to_\exists = \{(A, b, \{B, C\}), (C, t, \{D\})\}$,
- $\to_\forall = \{(B, b, \{C\})\}$,
- for all $s, v \in \{A, B, C, D\}$, $p_s \in V(v)$ iff $s = v$.

It is easy to verify that the last two correctness conditions are satisfied, e.g., for $A \xrightarrow{b}_{\exists} \{B, C\}$ we have $A \xrightarrow{b} B$. On the other hand, although $A \xrightarrow{t} D$, there is no $\xrightarrow{t}_{\exists}$ nor $\xrightarrow{t}_{\forall}$ from A to D.

Remark 1. Some readers may wonder whether further conditions on $\to_{\exists}$ and $\to_{\forall}$ should apply, to which we will come back in Section 4. For now, let us keep everything simple to understand the merit of the framework.

A fragment of EPDL is used to talk about the knowledge of a single agent on simple hyper models:

Definition 3 (Epistemic action language EAL). *Given a countable set of propositional variables* $\mathbf{P}$*, a finite sets of atomic actions* $\mathbf{\Sigma}$*, the formulas of EAL are given by:*

$$\phi \quad ::= \quad \top \mid p \mid \neg\phi \mid (\phi \wedge \phi) \mid K\phi \mid \langle a \rangle \phi$$

where $p \in \mathbf{P}$ *and* $a \in \mathbf{\Sigma}$.

As usual, we define $\bot$, $\phi \vee \psi$, $\phi \to \psi$, $\hat{K}\phi$ and $[a]\phi$ as the abbreviations of $\neg\top$, $\neg(\neg\phi \wedge \neg\psi)$, $\neg\phi \vee \psi$, $\neg K \neg\phi$ and $\neg\langle a \rangle \neg\phi$ respectively.

Definition 4 (Semantics). *The semantics of EAL on a simple hyper model* $\mathcal{M} = (S, \to, \to_{\exists}, \to_{\forall}, V)$ *is given by the following satisfaction relation w.r.t. a mode* $x \in \{0, \Box, \Diamond\}$*:*

$$\begin{array}{rcl}
\mathcal{M}, s \vDash \phi & \Leftrightarrow & \mathcal{M}, s \vDash_0 \phi \\
\mathcal{M}, s \vDash_x p & \Leftrightarrow & p \in V(s) \\
\mathcal{M}, s \vDash_x \phi \wedge \psi & \Leftrightarrow & \mathcal{M}, s \vDash_x \phi \text{ and } \mathcal{M}, s \vDash_x \psi \\
\mathcal{M}, s \vDash_x K\phi & \Leftrightarrow & \mathcal{M}, s \vDash_{\Box} \phi \\
\mathcal{M}, s \vDash_x \neg\phi & \Leftrightarrow & \begin{cases} \mathcal{M}, s \nvDash_0 \phi \text{ IF } x = 0 \\ \mathcal{M}, s \nvDash_{\Diamond} \phi \text{ IF } x = \Box \\ \mathcal{M}, s \nvDash_{\Box} \phi \text{ IF } x = \Diamond \end{cases} \\
\mathcal{M}, s \vDash_x \langle a \rangle \phi & \Leftrightarrow & \begin{cases} \exists t \in S : s \xrightarrow{a} t \text{ and } \mathcal{M}, t \vDash_0 \phi & \text{IF } x = 0 \\ \exists T \subseteq S : s \xrightarrow{a}_{\exists} T \text{ and } \forall t \in T : \mathcal{M}, t \vDash_{\Box} \phi & \text{IF } x = \Box \\ \forall T \subseteq S : s \xrightarrow{a}_{\forall} T \text{ implies } \exists t \in T : \mathcal{M}, t \vDash_{\Diamond} \phi & \text{IF } x = \Diamond \end{cases}
\end{array}$$

We say that ϕ *is valid in* $\mathcal{M}$ ($\mathcal{M} \vDash \phi$) *if for any* $s \in S_{\mathcal{M}}$: $\mathcal{M}, s \vDash \phi$. ϕ *is valid if for any* $\mathcal{M}$: $\mathcal{M} \vDash \phi$.

Clearly, this clumsy-looking semantics needs a good explanation. First of all, $\vDash_{\Box}$ and $\vDash_{\Diamond}$ are used as auxiliary semantics in order to define $\vDash$ ($\vDash_0$). 0, $\Box$ and $\Diamond$ can be viewed as *contexts* in evaluating the formulas. More precisely, 0 marks the factual mode: evaluating formulas outside the scope of any knowledge operator, while $\Box$ and $\Diamond$ denote the *knowledge modes* with the following intentions:

- $\vDash_{\Box} \phi$: the agent thinks that ϕ is *necessarily* true, i.e., ϕ is true in *all* the actual situations consistent with the procedural information that he has.
- $\vDash_{\Diamond} \phi$: the agent thinks that ϕ is *possibly* true, i.e., ϕ is true in *some* of the actual situations consistent with the procedural information that he has.

The alternations of $\Diamond$ and $\Box$ are triggered by negations: according to the agent, if ϕ is necessarily true then it is not possible to be not true, and if it is possible to be not true then it is not necessarily true. The clause for $K\phi$ says that the agent knows ϕ iff he thinks ϕ is necessarily true. Careful readers may wonder about the fact that $\mathcal{M}, s \vDash_0 p \iff \mathcal{M}, s \vDash_\Box p \iff \mathcal{M}, s \vDash_\Diamond p$, which means that a basic proposition is true iff the agent thinks that it is necessarily true iff the agent thinks that it is possibly true. This is because we assume the agent does not have any uncertainty about the basic propositions on the states. There is only uncertainty about the transitions ("*The agent knows all the cities but does not know how they are connected*").[1]

Note that the above semantics coincides with the standard possible world semantics on formulas without the K-operator. When evaluating epistemic formulas, things get more complicated. Note that we have a non-standard semantics for negation, thus it is worth working out the semantics for abbreviations, e.g., $\mathcal{M}, s \vDash_x \phi \to \psi$ may not be equivalent to $\mathcal{M}, s \vDash_x \phi$ implies $\mathcal{M}, s \vDash_x \psi$ depending on x. We summarize the results as follows (the readers are strongly encouraged to work out these by themselves):

$$
\begin{array}{rcl}
\mathcal{M}, s \vDash_x \phi \vee \psi & \Leftrightarrow & \mathcal{M}, s \vDash_x \phi \text{ or } \mathcal{M}, s \vDash_x \psi \\
\mathcal{M}, s \vDash_x \phi \to \psi & \Leftrightarrow & \begin{cases} \mathcal{M}, s \vDash_0 \phi \text{ implies } \mathcal{M}, s \vDash_0 \psi & \text{IF } x = 0 \\ \mathcal{M}, s \vDash_\Diamond \phi \text{ implies } \mathcal{M}, s \vDash_\Box \psi & \text{IF } x = \Box \\ \mathcal{M}, s \vDash_\Box \phi \text{ implies } \mathcal{M}, s \vDash_\Diamond \psi & \text{IF } x = \Diamond \end{cases} \\
\mathcal{M}, s \vDash_x \hat{K}\psi & \Leftrightarrow & \mathcal{M}, s \vDash_\Diamond \psi \\
\mathcal{M}, s \vDash_x [a]\phi & \Leftrightarrow & \begin{cases} \forall t : s \xrightarrow{a} t \text{ implies } \mathcal{M}, t \vDash_0 \phi & \text{IF } x = 0 \\ \exists T \subseteq S : s \xrightarrow{a}_\forall T \text{ and } \forall t \in T : \mathcal{M}, t \vDash_\Box \phi & \text{IF } x = \Box \\ \forall T \subseteq S : s \xrightarrow{a}_\exists T \text{ implies } \exists t \in T : \mathcal{M}, t \vDash_\Diamond \phi & \text{IF } x = \Diamond \end{cases}
\end{array}
$$

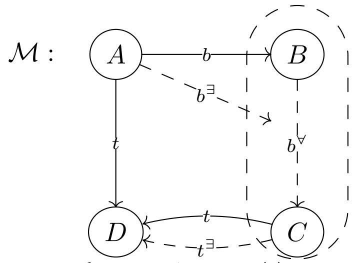
Now we see clearly that the agent knows ϕ iff ϕ is necessarily true to him, and ϕ is considered possible by the agent iff ϕ is possibly true to him. Let us also unravel the cases for $K\langle a\rangle\phi$ and $K[a]\phi$ to see the merit of the semantics more clearly:

$$
\begin{array}{rcl}
\mathcal{M}, s \vDash K\langle a\rangle\phi & \iff & \exists T \subseteq S : s \xrightarrow{a}_\exists T \text{ and } \forall t \in T : \mathcal{M}, t \vDash_\Box \phi \\
\mathcal{M}, s \vDash K[a]\phi & \iff & \exists T \subseteq S : s \xrightarrow{a}_\forall T \text{ and } \forall t \in T : \mathcal{M}, t \vDash_\Box \phi \\
\mathcal{M}, s \vDash \hat{K}\langle a\rangle\phi & \iff & \forall T \subseteq S : s \xrightarrow{a}_\forall T \text{ implies } \exists t \in T : \mathcal{M}, t \vDash_\Diamond \phi \\
\mathcal{M}, s \vDash \hat{K}[a]\phi & \iff & \forall T \subseteq S : s \xrightarrow{a}_\exists T \text{ implies } \exists t \in T : \mathcal{M}, t \vDash_\Diamond \phi
\end{array}
$$

The best way to understand the semantics is by looking at examples. Recall the model we mentioned earlier, we can verify the formulas on the right-hand

[1] Actually we can incorporate uncertainty about the basic propositions by adding over- and under-approximations of the truth values of them. We leave it to future work.

side:



$\mathcal{M} \vDash p_A \to \langle b \rangle (p_B \vee p_C)$
$\mathcal{M} \vDash p_B \to [b] p_C$
$\mathcal{M} \vDash p_C \to \langle t \rangle p_D$
$\mathcal{M}, A \vDash \langle t \rangle p_D \wedge \neg K \langle t \rangle p_D \wedge \neg K \neg \langle t \rangle p_D$
$\mathcal{M}, B \vDash [b] \neg p_D \wedge K [b] \neg p_D$
$\mathcal{M}, C \vDash \langle t \rangle p_D \wedge K \langle t \rangle p_D$
$\mathcal{M}, A \vDash K \langle b \rangle ((p_C \to \langle t \rangle p_D) \wedge (p_B \to [b] p_C))$

Let us take $\mathcal{M}, A \vDash \neg K \langle t \rangle p_D \wedge \neg K \neg \langle t \rangle p_D$ as an example:

$\mathcal{M}, A \vDash \neg K \langle t \rangle p_D \wedge \neg K \neg \langle t \rangle p_D$
$\iff \mathcal{M}, A \vDash_0 \neg K \langle t \rangle p_D \wedge \neg K \neg \langle t \rangle p_D$
$\iff \mathcal{M}, A \vDash_0 \neg K \langle t \rangle p_D$ and $\mathcal{M}, A \vDash_0 \neg K \neg \langle t \rangle p_D$
$\iff \mathcal{M}, A \nvDash_0 K \langle t \rangle p_D$ and $\mathcal{M}, A \nvDash_0 K \neg \langle t \rangle p_D$
$\iff \mathcal{M}, A \nvDash_\Box \langle t \rangle p_D$ and $\mathcal{M}, A \nvDash_\Box \neg \langle t \rangle p_D$
$\iff$ (not ($\exists T \subseteq S : A \xrightarrow{t}_\exists T$ and $\forall v \in T : \mathcal{M}, v \vDash_\Box p_D$)) and $\mathcal{M}, A \vDash_\Diamond \langle t \rangle p_D$
$\iff$ (it is not the case that $A \xrightarrow{t}_\exists \{D\}$) and ($\forall T \subseteq S : A \xrightarrow{t}_\forall T$ implies $\exists v \in T : \mathcal{M}, v \vDash_\Diamond p_D$)

Since there are no $\xrightarrow{t}_\exists$ nor $\xrightarrow{t}_\forall$ transitions from A, $\mathcal{M}, A \vDash \neg K \langle t \rangle p_D \wedge \neg K \neg \langle t \rangle p_D$.

In the above model $\mathcal{M}$, it seems that $\mathcal{M} \vDash K\phi \to \phi$, namely the knowledge is true. This is not accidental. We will show that the usual **S5** axioms are all valid. To prove it, we first show that if ϕ is necessarily true then it is true, and if it is true then it is possibly true.

Lemma 1. *For all the pointed simple hyper model $\mathcal{M}, s$, any ϕ the following two hold:*

1. $\mathcal{M}, s \vDash_\Box \phi$ *implies* $\mathcal{M}, s \vDash_0 \phi$
2. $\mathcal{M}, s \vDash_0 \phi$ *implies* $\mathcal{M}, s \vDash_\Diamond \phi$

Therefore $\mathcal{M}, s \vDash_\Box \phi$ *implies* $\mathcal{M}, s \vDash_\Diamond \phi$.

Proof. We prove the two claims simultaneously by induction on the structure of ϕ. $\phi = p$ or $\phi = \phi_1 \wedge \phi_2$: trivial. For $\phi = \neg\psi$: Suppose $\mathcal{M}, s \vDash_\Box \neg\psi$ then according to the semantics, $\mathcal{M}, s \nvDash_\Diamond \psi$ thus by IH (2nd claim), we have $\mathcal{M}, s \nvDash_0 \psi$ namely $\mathcal{M}, s \vDash_0 \neg\psi$. Similar for the second claim. For $\phi = K\psi$: Suppose $\mathcal{M}, s \vDash_\Box K\psi$ then $\mathcal{M}, s \vDash_\Box \psi$ thus according to the semantics $\mathcal{M}, s \vDash_0 K\psi$. Similarly, suppose $\mathcal{M}, s \vDash_0 K\psi$ then $\mathcal{M}, s \vDash_\Box \psi$, thus $\mathcal{M}, s \vDash_\Diamond K\psi$ according to the semantics.

For $\phi = \langle a \rangle \phi$: Suppose $\mathcal{M}, s \vDash_\Box \langle a \rangle \psi$ then there exists $T \subseteq S$ $s \xrightarrow{a}_\exists T$ and $T \vDash_\Box \psi$. According to the definition of hyper models, there is a $t \in T$ such that $s \xrightarrow{a} t$ and $\mathcal{M}, t \vDash_\Box \psi$. By IH we have there is a t such that $s \xrightarrow{a} t$ and $\mathcal{M}, t \vDash_0 \psi$ thus $\mathcal{M}, s \vDash_0 \langle a \rangle \psi$. Now for the second claim, suppose $\mathcal{M}, s \vDash_0 \langle a \rangle \psi$ then there is a t_0 such that $s \xrightarrow{a} t_0$ and $\mathcal{M}, t_0 \vDash_0 \psi$. By IH, $\mathcal{M}, t_0 \vDash_\Diamond \psi$. In order to show that $\mathcal{M}, s \vDash_\Diamond \langle a \rangle \psi$, we prove that for all $T \subseteq S : s \xrightarrow{a}_\forall T$ implies that there is a $t \in T$ such that $\mathcal{M}, t \vDash_\Diamond \psi$. Suppose not, then there exists a T_0 such that $s \xrightarrow{a}_\forall T_0$ and for all $t \in T_0 : \mathcal{M}, t \nvDash_\Diamond \psi$. Since $s \xrightarrow{a}_\forall T_0$, by the definition of hyper models we have for all $t : s \xrightarrow{a} t$ implies $t \in T_0$. Since $s \xrightarrow{a} t_0$, $t_0 \in T_0$ thus $\mathcal{M}, t_0 \nvDash_\Diamond \psi$, contradiction. Therefore for all $T \subseteq S : s \xrightarrow{a}_{\forall} T$ implies there is a t such that $\mathcal{M}, t \vDash_{\Diamond} \psi$, namely, $\mathcal{M}, s \vDash_{\Diamond} \langle a \rangle \psi$.

Theorem 1. *The following* ***S5*** *axiom schemas are valid:* `DIST` : $K(\phi \to \psi) \to (K\phi \to K\psi)$, `T` : $K\phi \to \phi$, `4` : $K\phi \to KK\phi$, `5` : $\neg K\phi \to K\neg K\phi$

Proof. For `DIST`:

$$\begin{aligned} & \mathcal{M}, s \vDash K(\phi \to \psi) \to (K\phi \to K\psi) \\ \iff & \mathcal{M}, s \vDash_0 K(\phi \to \psi) \text{ implies } \mathcal{M}, s \vDash_0 (K\phi \to K\psi) \\ \iff & \mathcal{M}, s \vDash_{\Box} \phi \to \psi \text{ implies} \\ & (\mathcal{M}, s \vDash_{\Box} \phi \text{ implies } \mathcal{M}, s \vDash_{\Box} \psi) \\ \iff & (\mathcal{M}, s \vDash_{\Diamond} \phi \text{ implies } \mathcal{M}, s \vDash_{\Box} \psi) \text{ implies} \\ & (\mathcal{M}, s \vDash_{\Box} \phi \text{ implies } \mathcal{M}, s \vDash_{\Box} \psi) \end{aligned}$$

Now suppose ($\mathcal{M}, s \vDash_{\Diamond} \phi$ implies $\mathcal{M}, s \vDash_{\Box} \psi$) and $\mathcal{M}, s \vDash_{\Box} \phi$. Since $\mathcal{M}, s \vDash_{\Box} \phi$ then by Theorem 1, we have $\mathcal{M}, s \vDash_{\Diamond} \phi$. Thus $\mathcal{M}, s \vDash_{\Box} \psi$.

For `T`:

$$\begin{aligned} & \mathcal{M}, s \vDash K\phi \to \phi \\ \iff & \mathcal{M}, s \vDash_0 K\phi \text{ implies } \mathcal{M}, s \vDash_0 \phi \\ \iff & \mathcal{M}, s \vDash_{\Box} \phi \text{ implies } \mathcal{M}, s \vDash_0 \phi \end{aligned}$$

From Theorem 1, it is clear that $\mathcal{M}, s \vDash K\phi \to \phi$.

For 4:

$$\begin{aligned} & \mathcal{M}, s \vDash K\phi \to KK\phi \\ \iff & \mathcal{M}, s \vDash_0 K\phi \text{ implies } \mathcal{M}, s \vDash_0 KK\phi \\ \iff & \mathcal{M}, s \vDash_{\Box} \phi \text{ implies } \mathcal{M}, s \vDash_{\Box} \phi \end{aligned}$$

For 5:

$$\begin{aligned} & \mathcal{M}, s \vDash \neg K\phi \to K\neg K\phi \\ \iff & \mathcal{M}, s \nvDash_0 K\phi \text{ implies } \mathcal{M}, s \vDash_0 K\neg K\phi \\ \iff & \mathcal{M}, s \nvDash_{\Box} \phi \text{ implies } \mathcal{M}, s \vDash_{\Box} \neg K\phi \\ \iff & \mathcal{M}, s \nvDash_{\Box} \phi \text{ implies } \mathcal{M}, s \nvDash_{\Diamond} K\phi \\ \iff & \mathcal{M}, s \nvDash_{\Box} \phi \text{ implies } \mathcal{M}, s \nvDash_{\Box} \phi \end{aligned}$$

Based on Lemma 1, we also have the following results:[2]

Proposition 1. *For any pointed hyper model $\mathcal{M}, s$ and any EAL formula ϕ the following hold:*

1. $\mathcal{M}, s \nvDash_{\Box} \phi \wedge \neg\phi$*, i.e.,* $\vDash_{\Box}$ *is consistent.*
2. $\mathcal{M}, s \vDash_{\Diamond} \phi \vee \neg\phi$*, i.e.,* $\vDash_{\Diamond}$ *is complete.*
3. $\vDash$ *is consistent and complete.*

Proof. Lemma 1 says that for any $\mathcal{M}, s$ any formula EAL ϕ: $\mathcal{M}, s \vDash_{\Box} \phi$ implies $\mathcal{M}, s \vDash_{\Diamond} \phi$.

For (1): Suppose $\mathcal{M}, s \vDash_{\Box} \neg\phi$ then $\mathcal{M}, s \nvDash_{\Diamond} \phi$ therefore $\mathcal{M}, s \nvDash_{\Box} \phi$. Thus $\mathcal{M}, s \nvDash_{\Box} \phi \wedge \neg\phi$.

For (2): ($\mathcal{M}, s \vDash_{\Diamond} \phi$ or $\mathcal{M}, s \vDash_{\Diamond} \neg\phi$) $\iff$ ($\mathcal{M}, s \vDash_{\Diamond} \phi$ or $\mathcal{M}, s \nvDash_{\Box} \phi$) $\iff$ ($\mathcal{M}, s \vDash_{\Box} \phi$ implies $\mathcal{M}, s \vDash_{\Diamond} \phi$). (3) is trivial by definition.

One way to interpret the above results is that the knowledge is consistent, and at least one of ϕ and $\neg\phi$ is considered possible by the agent. On the other hand,

[2] The use of the words *consistent* and *complete* are due to the convention in the abstraction literature c.f. e.g., [3].

$\vDash_\Box$ is not complete and $\vDash_\Diamond$ is not consistent, which can be demonstrated by the following simple example in which the agent has no information ($\xrightarrow{a}_\exists$ and $\xrightarrow{a}_\forall$ are empty):

$$s \xrightarrow{\quad a \quad} t$$

According to the semantics, $\mathcal{M}, s \vDash_\Diamond \langle a\rangle\top \wedge \neg\langle a\rangle\top$, and equivalently we have $\mathcal{M}, s \nvDash_\Box \neg\langle a\rangle\top$ and $\mathcal{M}, s \nvDash_\Box \langle a\rangle\top$. The "inconsistency" of $\vDash_\Diamond$ does not cause the inconsistency of $\vDash$ due to the semantics of the negation. Clearly, $\langle a\rangle\top \vee \neg\langle a\rangle\top$ is valid but $K(\langle a\rangle\top \vee \neg\langle a\rangle\top)$ is not valid in the above model, thus:

Proposition 2. *The rule of necessitation ($\vDash \phi$ infers $\vDash K\phi$) is not valid.*

On the other hand, our K operator is more *constructive* than the standard epistemic operator, demonstrated by the fact that K operator actually distributes over both $\vee$ and $\wedge$. In our setting, $K(\phi \vee \psi)$ should be read as 'the agent knows whether ϕ *or* ψ'. Correspondingly, $\hat{K}(\phi \wedge \psi)$ should be read as ' the agent considers both ϕ and ψ possible'.

Proposition 3. *The following are valid:*

$$\texttt{DIST}\wedge : K(\phi \wedge \psi) \leftrightarrow K\phi \wedge K\psi \quad \texttt{DIST}\vee : K(\phi \vee \psi) \leftrightarrow K\phi \vee K\psi$$

Since there is no uncertainty of basic propositions in the hyper models, the following holds:

Proposition 4. `INV` : $(p \to Kp) \wedge (\neg p \to K\neg p)$ *is valid.*

To completely axiomatize the logic, it is also important to have axioms controlling the interactions between K and $[a]$. Here we observe that the axiom of *perfect recall* (`PR`) is valid: $K[a]\phi \to [a]K\phi$ while the converse (*no learning*) is invalid.[3]

Proposition 5. `PR` : $K[a]\phi \to [a]K\phi$ *is valid.*

We leave it for future work whether `PR`, `INV`, `DIST`$\wedge$, `DIST`$\vee$, `DIST`, `T`, `5`, `4` on top of a propositional calculus are enough to completely axiomatize EAL over simple hyper frames.

3.2 Models with arbitrary procedural information

In this subsection, we consider arbitrary procedural information. Intuitively, the correct information $\langle \phi, \pi^\forall, \psi\rangle$ ($\langle \phi, \pi^\exists, \psi\rangle$) can be incorporated by adding to the model $\mathcal{M}$ a transition labelled by $\pi^\forall$ ($\pi^\exists$) from $\{\mathcal{M}, s \mid s \vDash \phi\}$ to $\{\mathcal{M}, t \mid t \vDash \psi\}$ and this leads to the definition of unrestricted hyper models (recall that $\Pi_{\mathbf{\Sigma}}$ is the set of regular expressions based on $\mathbf{\Sigma}$):

Definition 5 (Hyper model). *An hyper model is a tuple $(S, \to, \to_\exists, \to_\forall, V)$ where:*

[3] Versions of these axioms appear in temporal epistemic logic and dynamic epistemic logic (cf. [4,14]).

- $(S, \to, V)$ *is a Kripke model*
- $\to_\exists \subseteq 2^S \times \Pi_{\mathbf{\Sigma}} \times 2^S$ *is a labelled binary relation from a set of states to a set of states.*
- $\to_\forall \subseteq 2^S \times \Pi_{\mathbf{\Sigma}} \times 2^S$ *is a labelled binary relation from a set of states to a set of states.*
- *for all* $T, T' \subseteq S$*:* $T \xrightarrow{\pi}_\exists T'$ *implies that for all* $t \in T$ *there exists* $w \in \pi$ *and* $t' \in T'$ *such that* $t \xrightarrow{w} t'$*.*
- *for all* $T, T' \subseteq S$*:* $T \xrightarrow{\pi}_\forall T$ *implies that for all* $t \in T$ *all* $w \in \pi$*:* $t \xrightarrow{w} t'$ *implies* $t' \in T'$*.*

Again, the last two conditions guarantee that the information incorporated in the models are correct. The class of simple hyper models can be viewed as a subclass of hyper models, where the transitions are all in the shapes of $\{s\} \xrightarrow{a}_\exists T$ and $\{s\} \xrightarrow{a}_\forall T$.

Now we can consider the full EPDL language.

Definition 6 (Epistemic PDL). *Given a countable set of* propositional variables $\mathbf{P}$*, a finite set of* action symbols $\mathbf{\Sigma}$*, the formulas of EPDL language are defined by:*

$$\begin{array}{lll} \phi & ::= & \top \mid p \mid \neg\phi \mid (\phi \wedge \phi) \mid K\phi \mid \langle\pi\rangle\phi \\ \pi & ::= & a \mid \pi;\pi \mid \pi + \pi \mid \pi^* \end{array}$$

where $a \in \mathbf{\Sigma}$ *and* $p \in \mathbf{P}$*.*

Definition 7 (Semantics). *The semantics of EPDL on hyper models* $\mathcal{M} = (S, \to, \to_\exists, \to_\forall, V)$ *is defined similarly as the semantics of EAL on hyper models, with the following clauses replacing the clauses for* $\langle a\rangle\phi$ *formulas in the case of EAL:*

$$\begin{array}{l} \mathcal{M}, s \vDash_0 \langle\pi\rangle\phi \Leftrightarrow \exists t \in S : s \xrightarrow{w} t \text{ for some } w \in \pi \text{ such that } \mathcal{M}, t \vDash_0 \phi \\ \mathcal{M}, s \vDash_\Box \langle\pi\rangle\phi \Leftrightarrow \exists T_0, \dots, T_k \subseteq S, \exists \pi_1, \dots, \pi_k \in \Pi_{\mathbf{\Sigma}} : (s \in T_0, T_0 \xrightarrow{\pi_1}_\exists \cdots \xrightarrow{\pi_k}_\exists T_k, (\pi_1; \cdots; \pi_k) \subseteq \pi \\ \qquad \text{and } \forall t \in T_k : \mathcal{M}, t \vDash_\Box \phi) \\ \mathcal{M}, s \vDash_\Diamond \langle\pi\rangle\phi \Leftrightarrow \forall T_0, \dots, T_k \subseteq S, \forall \pi_1, \dots, \pi_k \in \Pi_{\mathbf{\Sigma}} : ((s \in T_0, T_0 \xrightarrow{\pi_1}_\forall \cdots \xrightarrow{\pi_k}_\forall T_k, \pi \subseteq (\pi_1; \cdots; \pi_k)) \\ \qquad \text{implies } (\exists t \in T_k : \mathcal{M}, t \vDash_\Diamond \phi)) \end{array}$$

It is not hard to see that on simple hyper models the above semantics coincides with the semantics of EAL on $\langle a\rangle\phi$ formulas. This justifies our abuse of $\vDash$ for both EPDL and EAL. The semantics says that the agent knows that $\langle\pi\rangle\phi$ at s if there is a 'refinement' of π ($(\pi_1; \cdots; \pi_k) \subseteq \pi$) such that each π_i step can be realized by a $\xrightarrow{\pi_i}_\exists$ transition, and in the end it will certainly reach a ϕ-state. Note that it is not necessary that $\pi_1; \cdots; \pi_k = \pi$, since we just need to guarantee there exists an execution of π.

We now prove the analogue of Lemma 1.

Lemma 2. *For all the pointed hyper model* $\mathcal{M}, s$*, any EPDL formula* ϕ *the following two hold: (1)* $\mathcal{M}, s \vDash_\Box \phi$ *implies* $\mathcal{M}, s \vDash_0 \phi$*; (2)* $\mathcal{M}, s \vDash_0 \phi$ *implies* $\mathcal{M}, s \vDash_\Diamond \phi$*. Therefore* $\mathcal{M}, s \vDash_\Box \phi$ *implies* $\mathcal{M}, s \vDash_\Diamond \phi$*.*

Proof. We only need to show the case of $\langle\pi\rangle\psi$. Suppose $\mathcal{M}, s \vDash_\Box \langle\pi\rangle\psi$ then:
$\exists T_0, \dots, T_k \subseteq S, \exists \pi_1, \dots, \pi_k \in \Pi : s \in T_0, T_0 \xrightarrow{\pi_1}_\exists \cdots \xrightarrow{\pi_k}_\exists T_k, \pi_1; \cdots; \pi_k \subseteq$

π and $\forall t \in T_k : \mathcal{M}, t \vDash_\Box \psi$.
Let $t_0 = s$. By the definition of hyper model, there exist $t_i \in T_i$ and $w_i \in \pi_i$ for $1 \leq i \leq k$ such that $t_{i-1} \stackrel{w_i}{\to} t_i$. It is clear that $w_1 \cdots w_k \in \pi_1; \cdots ; \pi_k$. Since $\pi_1; \cdots ; \pi_k \subseteq \pi$, $w_1 \cdots w_k \in \pi$. Thus by IH, $\mathcal{M}, s \vDash_0 \langle\pi\rangle\psi$.

Now for the second claim, suppose $\mathcal{M}, s \vDash_0 \langle\pi\rangle\psi$ then there is a t° such that $s \stackrel{w}{\to} t^\circ$ for a $w \in \pi$ and $\mathcal{M}, t^\circ \vDash_0 \psi$. By IH, $\mathcal{M}, t^\circ \vDash_\Diamond \psi$. Now suppose towards contradiction that $\mathcal{M}, s \nvDash_\Diamond \langle\pi\rangle\psi$ then according to the semantics we have:

$\exists T_0, \ldots, T_k \subseteq S, \exists \pi_1, \ldots, \pi_k \in \Pi : s \in T_0, T_0 \stackrel{\pi_1}{\to}_\forall \cdots \stackrel{\pi_k}{\to}_\forall T_k, \pi \subseteq \pi_1; \cdots ; \pi_k$ and $\forall t \in T_k : \mathcal{M}, t \nvDash_\Diamond \psi$

Obviously, if we can show that $t^\circ \in T_k$ then a contradiction is derived. In the following we prove that $t^\circ \in T_k$. Since $\pi \subseteq \pi_1; \cdots ; \pi_k$ and $w \in \pi$, $w \in \pi_1; \cdots ; \pi_k$. Therefore there exist $w_i \in \pi_i$ for $1 \leq i \leq k$ such that $w = w_1; \cdots ; w_k$ (w_i can be an empty string). According to the definition of hyper model, if $s \stackrel{w_1 \cdots w_i}{\to} t$ then $t \in T_i$ for all $1 \leq i \leq k$. In particular if $s \stackrel{w_1 \cdots w_k}{\to} t$ then $t \in T_k$. Now it is clear that $t^\circ \in T_k$.

Based on this lemma and the proof of Theorem 1, the following theorem holds immediately.

Theorem 2. `DIST`, `T`, `4`, *and* `5` *are valid for EPDL on hyper models.*

It is easy to verify that the EPDL analogies of Proposition 1 and Proposition 2 also hold.

4 Discussion and Future work

So far, we have only laid out the basics of an alternative semantics for EPDL based on hyper models where epistemic relations are replaced by two approximations of the actual transitions. In this section, we discuss some subtle issues about the semantics and point out further directions.

First of all, we justify that hyper models are indeed compact representations of a collection of Kripke models. On the one hand, each hyper model $\mathcal{M}$ can be unfolded into a set (call it *Unf*(M)) of Kripke models over the same set of states on which the imperfect information given by $\to_\exists$ and $\to_\forall$ transitions in the hyper model is correct, i.e., satisfying the last two conditions in the definition of hyper models. Based on Lemma 2, we can easily show that the knowledge in any hyper model $\mathcal{M}$ are truthful to *Unf*$(\mathcal{M})$, and $\mathcal{M}$ encodes all the possibilities in *Unf*$(\mathcal{M})$:

Proposition 6. *For every PDL formula ϕ and every s in any hyper model $\mathcal{M}$:*

- *if $\mathcal{M}, s \vDash K\phi$ then $\mathcal{N}, s \vDash \phi$ for every $\mathcal{N} \in$ Unf$(\mathcal{M})$*
- *if $\mathcal{N}, s \vDash \phi$ for some $\mathcal{N} \in$ Unf$(\mathcal{M})$ then $\mathcal{M}, s \vDash \hat{K}\phi$*

Note that even in very simple cases, $|$*Unf*$(\mathcal{M})|$ may be *exponential* in the size of the hyper model and $\mathbf{\Sigma}$. For example, let $S = \{s, t\}$, $V(p) = t$, and let $\pi_{\mathbf{\Sigma}}$ be the 'sum' of all actions in $\mathbf{\Sigma}$, then the hyper model with $\{s\} \stackrel{\pi_{\mathbf{\Sigma}}}{\to}_\forall \{t\}$ as the only transition has $2^{|\Sigma|}$ epistemically possible Kripke models to realize all the

$\phi_\Delta = \bigwedge_{a\in\Delta}\langle a\rangle p \wedge \bigwedge_{b\notin\Delta}\neg\langle b\rangle p$ formulas at s for each $\Delta \subseteq \mathbf{\Sigma}$. If the hyper model does not provide any procedural information (i.e., deleting the only transition) then $|\mathit{Unf}(\mathcal{M})| = 2^{|S|\cdot|\Sigma|\cdot|S|}$.

On the other hand, we may ask: is every set of concrete models (over a given set of states S) representable by a hyper model over S? Unfortunately, the answer is negative. For example, take the set of two Kripke models over $S = \{s, t\}$: $s \xrightarrow{a} t$ and $s \xrightarrow{b} t$. Let $\phi = (\langle a\rangle\top \wedge \neg\langle b\rangle\top) \vee (\langle b\rangle\top \wedge \neg\langle a\rangle\top)$. It is clear that $K\phi$ is true w.r.t. this set of models and state s. However, over S, no matter how the $\to_\forall$ and $\to_\exists$ transitions are chosen, we cannot make sure $K\phi$ holds on s since it would imply the knowledge of one of the disjunctions according to Proposition 3. The problem lies in the fact that we treat disjunction in the scope of K as "knowing whether" due to the semantics for Boolean connectives. We cannot really specify certain kinds of *inter-dependency* between the transitions. We may have a hyper model over s with the following transition $\{s\} \xrightarrow{(a+b)^\exists} \{t\}$, but we cannot make sure $\xrightarrow{a}$ and $\xrightarrow{b}$ are mutually exclusive between s and t. One potential solution is to make the labels of the transitions more expressive to specify conditional information, but we suspect a 'satisfactory' solution will in turn introduce another kind of 'state-explosion' on the transitions in hyper models. We need to find the balance between expressiveness and complexity.

Moreover, although the logic validates the `T` axiom, which makes sure everything we know is truthful, it is not very clear how much information we can 'know' by the semantics of EPDL on the hyper models. It seems that by requiring more conditions on the hyper model we may get more from the hyper models. Let us illustrate this in the simple hyper models. Below lists some intuitive closure properties that we may impose and their corresponding logical properties:

$(s \xrightarrow{a}_\forall T_1$ and $s \xrightarrow{a}_\forall T_2)$ implies $s \xrightarrow{a}_\forall T_1 \cap T_2$	$K[a]\phi \wedge K[a]\psi \to K[a](\phi \wedge \psi)$
$(s \xrightarrow{a}_\forall T_1$ and $s \xrightarrow{a}_\exists T_2)$ implies $s \xrightarrow{a}_\exists T_1 \cap T_2$	$K[a]\phi \wedge K\langle a\rangle\psi \to K\langle a\rangle(\phi \wedge \psi)$
for each s each a, there exists $T \subseteq S$ such that $s \xrightarrow{a}_\forall T$	$K[a]\top$

In fact, we only need one and only one $\xrightarrow{a}_\forall$ outgoing arrow from each state for each $a \in \mathbf{\Sigma}$ since all the $\xrightarrow{a}_\forall$ targeting sets can be intersected together, and the 'default' transition would be $s \xrightarrow{a}_\forall S$ where S is the set of all possible states. Note that, in general, the above requirements may help to bring back the missing reasoning power within the scope of K, due to the lack of necessitation rule for K in the logic.

To conclude, our epistemic framework is more compact and constructive compared to the standard possible-world approach of epistemic logic, in the sense that the hyper model resembles a collection of Kripke models and we can incrementally extend the model even from scratch by adding new imprecise information. At the same time, we pay the price that we are not able to represent all the collections of Kripke models since certain dependency of transitions is not encoded in the hyper models. To use the logic, we may make use of the 3-valued model checking algorithms (e.g., [6]), we leave out the exact complexity analysis to future work. Finally, it is also a natural next step to go probabilistic, as probabilities can be seen as another form of abstraction of qualitative information, as remarked in [8].

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