On Axiomatizations of Public Announcement Logic

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Abstract. In the literature, different axiomatizations of Public Announcement Logic (PAL) were proposed. Most of these axiomatizations share a 'core set' of the so-called reduction axioms. In particular, there is a composition axiom which stipulates how two consecutive announcements are composed into one. In this paper, by designing non-standard Kripke semantics for the language of PAL, we show that without the composition axiom the core set does not completely axiomatize PAL. In fact, most of the intuitive 'axioms' and rules we took for granted could not be derived from the core set. The non-standard semantics we proposed is of its own interest in modelling realistic agents. We show that with the help of different composition axioms we may axiomatize PAL w.r.t. such non-standard semantics.

1 Introduction

The language of *Public Announcement Logic (PAL)* [8,5] is usually presented as follows:

$$\phi ::= \top \mid p \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \Box_i\phi \mid [\psi]\phi$$

As usual, we define $\bot$, $\phi \vee \psi$, $\phi \to \psi$ and $\langle\psi\rangle\phi$ as the abbreviations of $\neg\top$, $\neg(\neg\phi \wedge \neg\psi)$, $\neg\phi \vee \psi$ and $\neg[\psi]\neg\phi$ respectively [1]. In this paper we call the $[\phi]$-free fragment of PAL language the language of *Epistemic Logic* (EL).

Given a Kripke model over a non-empty set of basic propositions $\mathbf{P}$, and a non-empty set of agents $\mathbf{I}$: $\mathcal{M} = (S, \{\to_i \mid i \in \mathbf{I}\}, V)$, the truth value of a PAL formula ϕ at a state s in $\mathcal{M}$ is defined as follows:

$$\begin{array}{rcl}
\mathcal{M}, s \vDash \top & \Leftrightarrow & \text{always} \\
\mathcal{M}, s \vDash p & \Leftrightarrow & p \in V(s) \\
\mathcal{M}, s \vDash \neg\phi & \Leftrightarrow & \mathcal{M}, s \nvDash \phi \\
\mathcal{M}, s \vDash \phi \wedge \psi & \Leftrightarrow & \mathcal{M}, s \vDash \phi \text{ and } \mathcal{M}, s \vDash \psi \\
\mathcal{M}, s \vDash \Box_i\psi & \Leftrightarrow & \forall t \rhd_i s : \mathcal{M}, t \vDash \psi \\
\mathcal{M}, s \vDash [\psi]\phi & \Leftrightarrow & \mathcal{M}, s \vDash \psi \text{ implies } \mathcal{M}|_\psi, s \vDash \phi
\end{array}$$

where $(\forall t \rhd_i s : \dots)$ denotes 'for all $t : s \to_i t$ implies $\dots$', and $\mathcal{M}|_\psi = (S', \{\to'_i \mid i \in \mathbf{I}\}, V')$ such that: $S' = \{s \mid \mathcal{M}, s \vDash \psi\}$, $\to'_i = \to_i |_{S' \times S'}$ and $V'(p) = V(p) \cap S'$. Note that in this paper we do not restrict ourselves to S5 model unless specified.

In the literature, different axiomatizations of PAL were proposed(cf. e.g.,[8], [1],[9],[10]). Most of these axiomatizations are based on the following proof system **PA**:

Axiom Schemas	
TAUT	all the instances of tautologies
DISTK	$\Box_i(\phi \to \psi) \to (\Box_i\phi \to \Box_i\psi)$
!ATOM	$[\psi]p \leftrightarrow (\psi \to p)$
!NEG	$[\psi]\neg\phi \leftrightarrow (\psi \to \neg[\psi]\phi)$
!CON	$[\psi](\phi \wedge \chi) \leftrightarrow ([\psi]\phi \wedge [\psi]\chi)$
!K	$[\psi]\Box_i\phi \leftrightarrow (\psi \to \Box_i(\psi \to [\psi]\phi))$
Rules	
NECK	$\dfrac{\phi}{\Box_i\phi}$
MP	$\dfrac{\phi, \phi \to \psi}{\psi}$

⋆ LORI-III (Springer LNCS Vol. 6953), .

[1] To simplify discussion, we do not consider common knowledge in this paper.

where ϕ, ψ, χ denote arbitrary formulas, p denotes an arbitrary propositional letter in $\mathbf{P}$ or $\top$, and i denotes an arbitrary index in $\mathbf{I}$ [2].

However, it is not clear whether the above system is complete. Moreover, based on the above system, there are different proposals for a complete system with extra axioms or rules. It is not clear either that, among those additional axioms and rules that are valid, which are necessary in the proof system of PAL, and which can be derived from **PA**[3]. In this paper, we will try to answer such questions. We first list the additional axiom schemas and rules that we will discuss in this paper as follows:

Axiom Schemas	
DIST!	$[\psi](\phi \to \chi) \to ([\psi]\phi \to [\psi]\chi)$
!COM	$[\psi][\chi]\phi \leftrightarrow [\psi \wedge [\psi]\chi]\phi$
WDIST!	$[\psi](\phi \to \chi) \leftrightarrow ([\psi]\phi \to (\psi \to [\psi]\chi))$
SDIST!	$[\psi](\phi \to \chi) \leftrightarrow ([\psi]\phi \to [\psi]\chi)$
!K′	$[\psi]\Box_i\phi \leftrightarrow (\psi \to \Box_i[\psi]\phi)$
EA!	$(\psi \to [\psi]\phi) \to [\psi]\phi$
DIA!	$(\psi \wedge [\psi]\phi) \leftrightarrow \langle\psi\rangle\phi$
Rules	
NEC!	$\dfrac{\phi}{[\psi]\phi}$
RE	$\dfrac{\phi \leftrightarrow \chi}{\psi \leftrightarrow \psi[\chi/\phi]}$
RE¬	$\dfrac{\phi \leftrightarrow \chi}{\neg\phi \leftrightarrow \neg\chi}$
RE∧	$\dfrac{\phi \leftrightarrow \chi}{(\psi \wedge \phi) \leftrightarrow (\psi \wedge \chi)}$
RE□	$\dfrac{\phi \leftrightarrow \chi}{\Box_i\phi \leftrightarrow \Box_i\chi}$
RE!	$\dfrac{\phi \leftrightarrow \chi}{[\psi]\phi \leftrightarrow [\psi]\chi}$

The selection of these axioms and rules is not arbitrary: RE is the rule of *replacement of equivalents* where $\psi[\chi/\phi]$ denotes any formula obtained by replacing one or more occurrences of ϕ in ψ with χ (cf. e.g., the axiomatization in [8]). !COM is the *composition axiom* featured in many expositions of PAL proof system (cf. e.g., [10]). Other usual suspects in the above table include !K′, NEC! and DIST!. !K′ is often used in the literature as an "equivalent" of !K (cf., e.g., [9]), while NEC! and DIST! are $[\phi]$-versions of the well-known necessitation rule and distribution axiom in basic modal logic, which sometimes appear too in the axiomatizations of PAL (cf. e.g. [1]). WDIST! and SDIST! are (weaker/stronger) variations of DIST! that we will use. Finally EA! and DIA! are usually taken for granted but EA! does play an important role in our later discussions.

In the literature, it is shown that the following systems are sound and complete w.r.t. the standard semantics of PAL: **PA** + !COM (cf. e.g., [10]), **PA** + RE (cf. [8]).

The main technical contributions of this paper can be summarized as follows:

- WDIST!, DIA!, RE¬, RE∧, and RE□ can be derived in **PA**.
- None of !COM, NEC!, RE!, RE, DIST!, SDIST!, EA!, and !K′ can be derived in **PA**.
- **PA**, **PA** + !K′ + EA! + DIST!, **PA** + NEC! are not complete w.r.t. the standard PAL semantics while **PA** + DIST! + NEC! is complete.
- To prove the above results, we introduce two non-standard semantics of PAL such that both validate **PA**. Moreover, we show **PA** plus a variation of the composition axiom $[\psi][\chi]\phi \leftrightarrow [\psi \wedge \chi]\phi$)(!COM∧) is sound and complete w.r.t one of the non-standard semantics. Under this semantics PAL is still equivalent to EL qua expressive power.

[2] Note that **PA** does not include the rule of uniform substitution (US). A discussion on the decidability of the US-closed fragment of PAL can be found in [6].

[3] A rule $\frac{\phi}{\psi}$ is derivable from a system **S** if all the instances of this rule can be derived by using ϕ, axioms and inference rules in **S**.

The points we would like to make are as follows:

- Axiomatizating PAL and other dynamic epistemic logics (DEL) is more subtle than it may look, which invites careful investigations.
- There are two general ways to conduct the reductions from DEL to the base logic EL: 'inside-out' (by using RE) and 'outside-in' (by using composition axioms). Composition axioms give explicit information about the dynamics of the system. Various forms of composition axioms can be used in axiomatizing DEL under non-standard semantics when RE rule is not be valid.
- The development of DEL is so far in principal semantics-driven, however, the syntactic perspective may bring new insights. Other useful variations of DEL can be designed for good reasons e.g., for modelling richer phenomenons regarding interactions between updates.

The paper is organized as follows: In Section 2, we review some known results about the axiomatizations of **PA** and make some useful observations linking many axioms and rules. In Section 3, we show that !COM, NEC!, RE!, RE, DIST!, EA!, !K$'$ are not derivable from **PA** by giving two non-standard semantics which validate **PA** but invalidate (all or some of) the above mentioned axioms and rules, thus showing that **PA** is not complete w.r.t. the standard PAL semantics. Moreover, we prove that by adding the right composition axiom we can obtain a complete system w.r.t. the first non-standard semantics which intuitively models a 'slow' agent. Finally we conclude with discussions and future work in Section 4.

2 Preliminaries

Proposition 1. **PA** *is sound w.r.t to the standard PAL semantics.*

PROOF Cf. e.g., [10]. □

Moreover, it is an easy exercise to show that all the other axioms and rules mentioned in the introduction are valid w.r.t. to the standard PAL semantics:

Proposition 2. *Axiom schemas* DIST!, !COM, WDIST!, SDIST!, !K$'$, EA!, DIA!, *and inference rules* NEC!, RE, RE$\neg$, RE$\wedge$, RE$\Box$, RE! *are all valid w.r.t. the standard PAL semantics.*

A natural question to ask is: are they derivable in **PA**? We list a few positive answers here.

Proposition 3. RE$\neg$, RE$\wedge$, RE$\Box$ *can be derived in* **PA**.

PROOF RE$\neg$, RE$\wedge$ are trivial by using TAUT. Here we only show the (standard) reasoning behind RE$\Box$.

$1 \vdash_{\mathbf{PA}} \phi \leftrightarrow \chi$	
$2 \vdash_{\mathbf{PA}} \phi \to \chi$	TAUT
$3 \vdash_{\mathbf{PA}} \Box_i(\phi \to \chi)$	NECK
$4 \vdash_{\mathbf{PA}} \Box_i(\phi \to \chi) \to (\Box_i\phi \to \Box_i\chi)$	DISTK
$5 \vdash_{\mathbf{PA}} \Box_i\phi \to \Box_i\chi$	MP(3, 4)
$6 \vdash_{\mathbf{PA}} \Box_i\phi \leftrightarrow \Box_i\chi$	repeat 2-5 for $\chi \to \phi$, TAUT

□

Note that the above proof uses NECK and DISTK, however, as we will see in Section 3 the $[\phi]$ versions of them (DIST! and NEC!) can not be derived in **PA**.

Based on the above proposition, we know that the following restricted version of RE holds.

Proposition 4. *The following rule* RE^r *is valid: Given* $\vdash_{\mathbf{PA}} \phi \leftrightarrow \chi$, *we have* $\vdash_{\mathbf{PA}} \psi \leftrightarrow \psi'$ *where* ψ' *is obtained by replacing some occurrences of* ϕ *in* ψ *with* χ, *provided that these occurrences of* ϕ *are "good", i.e., they do not appear in the scope of any announcement operator.*

Proof Suppose $\vdash_{\mathbf{PA}} \phi \leftrightarrow \chi$ and ψ' is obtained from ψ by replacing some "good" occurrences of ϕ by χ. It is not hard to see that we can construct ψ and ψ' from ϕ, χ and other formulas by using the equivalence preserving operations: $\mathtt{RE}\neg$,$\mathtt{RE}\wedge$,$\mathtt{RE}\Box$. For example, from $\vdash_{\mathbf{PA}} \phi \leftrightarrow \chi$ we can show that $\vdash_{\mathbf{PA}} ([\psi]\phi \to \Box_i\phi) \leftrightarrow ([\psi]\phi \to \Box_i\chi)$ by taking $[\psi]\phi$ as one of the atomic build blocks.

□

Proposition 5. DIA! *is a theorem schema of* **PA**.

Proof

1	$\vdash_{\mathbf{PA}} [\psi]\neg\phi \leftrightarrow (\psi \to \neg[\psi]\phi)$	!NEG
2	$\vdash_{\mathbf{PA}} \neg[\psi]\neg\phi \leftrightarrow \neg(\psi \to \neg[\psi]\phi)$	$\mathtt{RE}\neg(1)$
3	$\vdash_{\mathbf{PA}} \neg(\psi \to \neg[\psi]\phi) \leftrightarrow (\psi \wedge [\psi]\phi)$	TAUT
4	$\vdash_{\mathbf{PA}} \neg[\psi]\neg\phi \leftrightarrow (\psi \wedge [\psi]\phi)$	$\mathtt{RE}^r(2,3)$

□

Proposition 6. WDIST! *is a theorem schema of* **PA**.

Proof Note that $\phi \to \chi$ is the abbreviation of $\neg(\phi \wedge \neg\chi)$. Thus $[\psi](\phi \to \chi)$ is the abbreviation of $[\psi]\neg(\phi \wedge \neg\chi)$.

1	$\vdash_{\mathbf{PA}} [\psi]\neg(\phi \wedge \neg\chi) \leftrightarrow (\psi \to \neg[\psi](\phi \wedge \neg\chi))$	!NEG
2	$\vdash_{\mathbf{PA}} [\psi](\phi \wedge \neg\chi) \leftrightarrow ([\psi]\phi \wedge [\psi]\neg\chi)$	!CON
3	$\vdash_{\mathbf{PA}} \neg[\psi](\phi \wedge \neg\chi) \leftrightarrow \neg([\psi]\phi \wedge [\psi]\neg\chi)$	$\mathtt{RE}\neg(2)$
4	$\vdash_{\mathbf{PA}} [\psi]\neg\chi \leftrightarrow (\psi \to \neg[\psi]\chi)$	!NEG
5	$\vdash_{\mathbf{PA}} \neg[\psi](\phi \wedge \neg\chi) \leftrightarrow \neg([\psi]\phi \wedge (\psi \to \neg[\psi]\chi)$	$\mathtt{RE}^r(3,4)$
6	$\vdash_{\mathbf{PA}} [\psi](\phi \to \chi) \leftrightarrow (\psi \to \neg([\psi]\phi \wedge (\psi \to \neg[\psi]\chi))$	$\mathtt{RE}^r(5,1)$
7	$\vdash_{\mathbf{PA}} [\psi](\phi \to \chi) \leftrightarrow (\psi \to ([\psi]\phi \to (\psi \wedge [\psi]\chi)))$	TAUT
8	$\vdash_{\mathbf{PA}} [\psi](\phi \to \chi) \leftrightarrow ((\psi \wedge [\psi]\phi) \to (\psi \wedge [\psi]\chi))$	TAUT
9	$\vdash_{\mathbf{PA}} [\psi](\phi \to \chi) \leftrightarrow ((\psi \wedge [\psi]\phi) \to [\psi]\chi)$	TAUT
10	$\vdash_{\mathbf{PA}} [\psi](\phi \to \chi) \leftrightarrow ([\psi]\phi \to (\psi \to [\psi]\chi))$	TAUT

□

Note that $\vdash_{\mathbf{PA}} [\psi]\chi \to (\psi \to [\psi]\chi)$, thus if $\vdash_{\mathbf{PA}}$ EA! then $\vdash_{\mathbf{PA}} [\psi]\chi \leftrightarrow (\psi \to [\psi]\chi)$. Based on this observation and $\mathtt{RE}^r$, it is clear that if $\vdash_{\mathbf{PA}}$ EA! then SDIST! (and DIST!) can be proved in **PA**. However, $\nvdash_{\mathbf{PA}}$ EA! as we will see in Section 3.

If we extend **PA** with EA! and NEC!, then RE is derivable.

Proposition 7. RE! *and* RE *are derivable in* **PA**+EA!+NEC! *and* **PA**+DIST!+NEC!.

Proof $\vdash_{\mathbf{PA}+\mathtt{EA!}}$ DIST!, together with NEC! we can derive RE! (cf. the proof of Proposition 3). Then based on the proof of Proposition 4, RE can be derived. □

Proposition 8. EA! *is a theorem schema of* **PA**+!COM.

Proof By induction on the structure of ϕ (cf. [10, pp.251]). □

By using the reduction axioms in **PA** and the above restricted substitution rule we can translate most of PAL formulas to equivalent EL formulas by iteratively replacing the inner part of the formula with an equivalent announcement-free formula. However formulas in the shape of $[\psi][\chi]\phi$ may be problematic since RE! is missing in **PA**.

Here we mention a few completeness results by using such reductions.

Theorem 1. **PA**+!COM *is sound and (weakly) complete w.r.t. the standard semantics of* PAL.

PROOF We only sketch the proof in [10][4]. We first define a translation $t : \texttt{PAL} \to \texttt{EL}$:

$$\begin{array}{llll} t(\top) & = \top & t([\psi]\top) & = t(\psi \to \top) \\ t(p) & = p & t([\psi]p) & = t(\psi \to p) \\ t(\neg\phi) & = \neg t(\phi) & t([\psi]\neg\phi) & = t(\psi \to \neg[\psi]\phi) \\ t(\phi_1 \wedge \phi_2) & = t(\phi_1) \wedge t(\phi_2) & t([\psi](\phi_1 \wedge \phi_2)) & = t([\psi]\phi_1 \wedge [\psi]\phi_2) \\ t(\Box_i\phi) & = \Box_i t(\phi) & t([\psi]\Box_i\phi) & = t(\psi \to \Box_i(\psi \to [\psi]\phi)) \\ & & t([\psi][\chi]\phi) & = t([\psi \wedge [\psi]\chi]\phi) \end{array}$$

Based on a suitable definition of the complexity of formulas (cf. [10]) we can show that the translation/rewriting always reduce the complexity thus it will terminate at some point. Note that in the process of the rewriting, $t(\psi)$ never falls in the scope of any announcement operator. Based on this observation, by induction on the complexity of the formulas we can show that $\vdash_{\mathbf{PA}+!\texttt{COM}} \phi \leftrightarrow t(\phi)$ (using reduction axioms and $\texttt{RE}\wedge$, $\texttt{RE}\neg$, and $\texttt{RE}\Box$). By soundness of $\mathbf{PA}+!\texttt{COM}$, $\vDash \phi \leftrightarrow t(\phi)$. Now suppose $\vDash \phi$ then $\vDash t(\phi)$. Thus by the completeness of the basic modal logic $\mathbf{K}$, $\vdash_{\mathbf{K}} t(\phi)$. Therefore $\vdash_{\mathbf{PA}+!\texttt{COM}} t(\phi)$. Since $\vdash_{\mathbf{PA}+!\texttt{COM}} \phi \leftrightarrow t(\phi)$, we have $\vdash_{\mathbf{PA}+!\texttt{COM}} \phi$ by MP. □

Theorem 2 ([8]). PA+RE *is sound and weakly complete w.r.t. the standard semantics of PAL.*

PROOF Similar to the above proof, we only need to revise the last item of the translation function t as follows:

$$t([\psi][\chi]\phi) = t([\psi]t([\chi]\phi))$$

Note that now we do need the full power of RE since t does fall in the scope of announcement operators. □

As a straight forward corollary, we have:

Corollary 1. PA+DIST!+NEC! *and* **PA+EA!+NEC!** *are sound and complete w.r.t. the standard semantics of PAL.*

PROOF From Proposition 7 and the above theorem. □

Note that the translation of $[\psi][\chi]\phi$ formulas defined in Theorem 2 is in the fashion of 'inside-out' while the translation in Theorem 1 is 'outside-in'.

3 PA is not complete

In this section we give two alternative semantics for the language of PAL which validate **PA** but make many intuitive axioms and rules invalid.

3.1 A context-dependent semantics

Inspired by the semantics developed in [4,11,2], we define the satisfaction relation w.r.t. a *context* ρ (notation: $\vDash_\rho$), which is used to record the information from previous announcements.

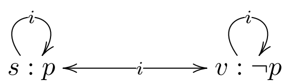
Given a Kripke model over $\mathbf{P}, \mathbf{I}$: $\mathcal{M} = (S, \{\to_i \mid i \in \mathbf{I}\}, V)$, the truth value of a PAL formula ϕ at a state s in $\mathcal{M}$ is recursively defined as based on $\Vdash_\rho$ where ρ is a formula in the language of PAL:

$$\boxed{\begin{array}{rcl} \mathcal{M}, s \Vdash \phi & \Leftrightarrow & \mathcal{M}, s \Vdash_\top \phi \\ \mathcal{M}, s \Vdash_\rho \top & \Leftrightarrow & \text{always} \\ \mathcal{M}, s \Vdash_\rho p & \Leftrightarrow & p \in V(s) \\ \mathcal{M}, s \Vdash_\rho \neg\phi & \Leftrightarrow & \mathcal{M}, s \nVdash_\rho \phi \\ \mathcal{M}, s \Vdash_\rho \phi \wedge \psi & \Leftrightarrow & \mathcal{M}, s \Vdash_\rho \phi \text{ and } \mathcal{M}, s \Vdash_\rho \psi \\ \mathcal{M}, s \Vdash_\rho \Box_i\phi & \Leftrightarrow & \forall t \rhd_i s : \mathcal{M}, t \Vdash_\top \rho \text{ implies } \mathcal{M}, t \Vdash_\rho \phi \\ \mathcal{M}, s \Vdash_\rho [\psi]\phi & \Leftrightarrow & \mathcal{M}, s \Vdash_\top \psi \text{ implies } \mathcal{M}, s \Vdash_{\rho\wedge\psi} \phi \end{array}}$$

[4] We need to adapt the proof just a little bit to fit !K in the proof instead of !K′ used in [10].

Note that instead of updating the model we somehow remember the announcements and recall them as the context only in evaluating $\Box_i$ formulas. Remembering the context is an alternative way of doing model relativization. We say that ϕ is *valid* w.r.t. this non-standard semantics if $\Vdash \phi$ (i.e. $\Vdash_\top \phi$).

Example 1. Consider the following (S5) model $\mathcal{M}$ with two worlds s, v:



$\mathcal{M}, s \Vdash \neg\Box_i p \iff \mathcal{M}, s \nVdash_\top \Box_i p \iff (\exists t \rhd_i s : \mathcal{M}, t \Vdash_\top \top \text{ and } \mathcal{M}, t \nVdash_\top p)$
Since $p \notin V(v)$ and $s \xrightarrow{i} v$, $\mathcal{M}, s \Vdash \neg\Box_i p$.

$\mathcal{M}, s \Vdash_p \Box_i p \iff (\forall t \rhd_i s : \mathcal{M}, t \Vdash_\top p \text{ implies } \mathcal{M}, t \Vdash_p p)$. Clearly, $\mathcal{M}, s \Vdash_p \Box_i p$. Similarly $\mathcal{M}, s \Vdash_{\top\wedge p} \Box_i p$.

$\mathcal{M}, s \Vdash [p]\Box_i p \iff (\mathcal{M}, s \Vdash_\top p \text{ implies } \mathcal{M}, s \Vdash_{\top\wedge p} \Box_i p)$
$\iff \mathcal{M}, s \Vdash_{\top\wedge p} \Box_i p$. Thus, $\mathcal{M}, s \Vdash [p]\Box_i p$ (based on the above example).

$\mathcal{M}, s \Vdash [p \wedge \neg\Box_i p]\Box_i p \iff (\mathcal{M}, s \Vdash_\top p \wedge \neg\Box_i p \text{ implies } \mathcal{M}, s \Vdash_{\top\wedge p\wedge\neg\Box_i p} \Box_i p)$
$\iff (\mathcal{M}, s \Vdash_\top p \text{ and } \mathcal{M}, s \Vdash_\top \neg\Box_i p) \text{ implies } \mathcal{M}, s \Vdash_{\top\wedge p\wedge\neg\Box_i p} \Box_i p$
$\iff \mathcal{M}, s \Vdash_{\top\wedge p\wedge\neg\Box_i p} \Box_i p$ (from the above examples)
$\iff \forall t \rhd_i s : \mathcal{M}, t \Vdash_\top \top \wedge p \wedge \neg\Box_i p \text{ implies } \mathcal{M}, t \Vdash_{\top\wedge p\wedge\neg\Box_i p} p$.
It is easy to see that $\mathcal{M}, s \Vdash [p \wedge \neg\Box_i p]\Box_i p$.

$\mathcal{M}, s \Vdash [p][\neg\Box_i p]\bot \iff (\mathcal{M}, s \Vdash_\top p \text{ implies } \mathcal{M}, s \Vdash_{\top\wedge p} [\neg\Box_i p]\bot)$
$\iff \mathcal{M}, s \Vdash_\top \neg\Box_i p \text{ implies } \mathcal{M}, s \Vdash_{\top\wedge p\wedge\neg\Box_i p} \bot$. Thus $\mathcal{M}, s \nVdash [p][\neg\Box_i p]\bot$.
On the other hand, it is easy to verify that $\mathcal{M}, s \vDash [p][\neg\Box_i p]\bot$ (recall that $\vDash$ denotes the standard semantics).

♣

In the example, it seems that $\Vdash$ coincides with $\vDash$ except for the formulas with consecutive announcements. We will show that it is not a coincidence.

Proposition 9. *$\Vdash$ coincides with $\vDash$ on* `EL` *formulas.*

PROOF Note that without $[\psi]$ operators, ρ can never be changed to any non-trivial formula during the evaluation of a formula. Since $\mathcal{M}, s \Vdash_\top \top$ is always true, it is easy to see that the definition of $\Vdash_\top$ coincides with $\vDash$ for `EL` formulas. □

Before going further we first prove two useful propositions. Let `!COM`$\wedge$ be the axiom schema $[\psi][\chi]\phi \leftrightarrow [\psi \wedge \chi]\phi$ which differs from `!COM`.

Proposition 10. `!COM`$\wedge$ *is valid w.r.t.* $\Vdash$.

PROOF $\mathcal{M}, s \Vdash [\psi][\chi]\phi \iff \mathcal{M}, s \Vdash_\top \psi \text{ implies } \mathcal{M}, s \Vdash_{\top\wedge\psi} [\chi]\phi$
$\iff \mathcal{M}, s \Vdash_\top \psi \text{ implies } (\mathcal{M}, s \Vdash_\top \chi \text{ implies } \mathcal{M}, s \Vdash_{\top\wedge\psi\wedge\chi} \phi)$
$\iff (\mathcal{M}, s \Vdash_\top \psi \text{ and } \mathcal{M}, s \Vdash_\top \chi) \text{ implies } \mathcal{M}, s \Vdash_{\top\wedge\psi\wedge\chi} \phi$
$\iff (\mathcal{M}, s \Vdash_\top \psi \wedge \chi) \text{ implies } \mathcal{M}, s \Vdash_{\top\wedge\psi\wedge\chi} \phi$
$\iff \mathcal{M}, s \Vdash [\psi \wedge \chi]\phi$ □

Proposition 11. *For any* `PAL` *formulas χ, ψ, and ϕ : if $\Vdash \chi \leftrightarrow \psi$ then for all pointed model $\mathcal{M}, s$: $\mathcal{M}, s \Vdash_\chi \phi \iff \mathcal{M}, s \Vdash_\psi \phi$. As a consequence, the following rule is valid:*

$$\texttt{!RE} : \frac{\phi \leftrightarrow \chi}{[\phi]\psi \leftrightarrow [\chi]\psi}$$

PROOF   By induction on the structure of ϕ. The Boolean cases are trivial. Now let $\phi = \Box_i\phi'$. Note that $\mathcal{M}, s \Vdash_\rho \Box_i\phi' \iff \forall t \rhd_i s : \mathcal{M}, t \Vdash_\top \rho$ implies $\mathcal{M}, t \Vdash_\rho \phi'$. Since $\Vdash \chi \leftrightarrow \psi$, for all $\mathcal{M}, t$: $\mathcal{M}, t \Vdash_\top \chi \iff \mathcal{M}, t \Vdash_\top \psi$. Therefore, based on the induction hypothesis that $\mathcal{M}, t \Vdash_\chi \phi' \iff \mathcal{M}, t \Vdash_\psi \phi'$, $\mathcal{M}, s \Vdash_\chi \Box_i\phi \iff \mathcal{M}, s \Vdash_\psi \Box_i\phi$.

Now consider $\phi = [\phi']\phi''$. According to the semantics of conjunctions, it is not hard to see that if $\Vdash \psi \leftrightarrow \chi$ then for any ϕ' we have $\Vdash (\psi \wedge \phi') \leftrightarrow (\chi \wedge \phi')$. Now according to the truth condition of $[\phi']\phi''$ and induction hypothesis, $\mathcal{M}, s \Vdash_\chi [\phi']\phi'' \iff \mathcal{M}, s \Vdash_\psi [\phi']\phi''$. Based on the these observations, it is easy to show that $\Vdash \psi \leftrightarrow \chi$ implies $\Vdash [\chi]\phi \leftrightarrow [\psi]\phi$. □

Remark 1. The admissible rule `!RE` is itself interesting in axiomatizing `PAL`. We conjecture that it is not derivable from **PA** but leave it for future work.

In the following we show that **PA** is sound w.r.t. $\Vdash$. Actually many other rules and axiom schemas are also valid under $\Vdash$ as we will see soon.

Lemma 1. *`TAUT`, `MP`, `NECK`, and `DISTK` are valid w.r.t. $\Vdash$.*

PROOF   For `TAUT` and `MP`: Trivial (check the truth conditions for Boolean cases).
For `NECK`: Suppose $\Vdash \phi$ then for all models $\mathcal{M}, s \Vdash_\top \phi$. Suppose towards a contradiction that there is a model $\mathcal{M}, s \Vdash_\top \neg\Box_i\phi$. According to the semantics there exists $t \rhd_i s$ $\mathcal{M}, t \Vdash_\top \top$ and $\mathcal{M}, t \Vdash_{\top\wedge\top} \neg\phi$, contradiction.

For `DISTK`: Suppose $\mathcal{M}, s \Vdash \Box_i(\phi \to \psi)$ then for all $t \rhd_i s$ $\mathcal{M}, t \Vdash_\top \phi \to \psi$. Now suppose $\mathcal{M}, s \Vdash \Box_i\phi$ then for all $t \rhd_i s$: $\mathcal{M}, t \Vdash_\top \top$ implies $\mathcal{M}, t \Vdash_\top \phi$. It is clear that for all $t \rhd_i s$: $\mathcal{M}, t \Vdash_\top \top$ implies $\mathcal{M}, t \Vdash_\top \psi$. Thus $\mathcal{M}, s \Vdash \Box_i\psi$. Therefore $\mathcal{M}, s \Vdash \Box_i(\phi \to \psi) \to (\Box_i\phi \to \Box_i\psi)$. □

Lemma 2. *`!ATOM`, `!NEG`, `!CON`, `!K`, `!K'`, and `EA!` are valid w.r.t. $\Vdash$.*

PROOF   For `!ATOM`: $\mathcal{M}, s \Vdash [\psi]p \iff (\mathcal{M}, s \Vdash_\top \psi$ implies $\mathcal{M}, s \Vdash_{\top\wedge\psi} p)$
$\iff (\mathcal{M}, s \Vdash_\top \psi$ implies $\mathcal{M}, s \Vdash_\top p) \iff \mathcal{M}, s \Vdash \psi \to p$.

For `!NEG`: $\mathcal{M}, s \Vdash [\psi]\neg\phi \iff (\mathcal{M}, s \Vdash_\top \psi$ implies $\mathcal{M}, s \Vdash_{\top\wedge\psi} \neg\phi)$ while $\mathcal{M}, s \Vdash \psi \to \neg[\psi]\phi \iff (\mathcal{M}, s \Vdash_\top \psi$ implies $\mathcal{M}, s \Vdash_\top \neg[\psi]\phi)$
$\iff \mathcal{M}, s \Vdash_\top \psi$ implies $(\mathcal{M}, s \Vdash_\top \psi$ and $\mathcal{M}, s \Vdash_{\top\wedge\psi} \neg\phi)$
$\iff \mathcal{M}, s \Vdash_\top \psi$ implies $\mathcal{M}, s \Vdash_{\top\wedge\psi} \neg\phi$. Thus $\mathcal{M}, s \Vdash [\psi]\neg\phi \leftrightarrow (\psi \to \neg[\psi]\phi)$.

For `!CON`: $\mathcal{M}, s \Vdash [\psi](\phi \wedge \chi) \iff (\mathcal{M}, s \Vdash_\top \psi$ implies $\mathcal{M}, s \Vdash_{\top\wedge\psi} \phi \wedge \chi)$
$\iff (\mathcal{M}, s \Vdash_\top \psi$ implies $\mathcal{M}, s \Vdash_{\top\wedge\psi} \phi)$ and $(\mathcal{M}, s \Vdash_\top \psi$ implies $\mathcal{M}, s \Vdash_{\top\wedge\psi} \chi) \iff \mathcal{M}, s \Vdash_\top [\psi]\phi \wedge [\psi]\chi$.

For `!K`: $\mathcal{M}, s \Vdash [\psi]\Box_i\phi \iff \mathcal{M}, s \Vdash_\top \psi$ implies $\mathcal{M}, s \Vdash_{\top\wedge\psi} \Box_i\phi$ while $\mathcal{M}, s \Vdash \psi \to \Box_i(\psi \to [\psi]\phi) \iff \mathcal{M}, s \Vdash_\top \psi$ implies $\mathcal{M}, s \Vdash_\top \Box_i(\psi \to [\psi]\phi)$
$\iff \mathcal{M}, s \Vdash_\top \psi$ implies $(\forall t \rhd s : \mathcal{M}, t \Vdash_\top \top$ implies $(\mathcal{M}, t \Vdash_\top \psi$ implies $\mathcal{M}, t \Vdash_\top [\psi]\phi))$
$\iff \mathcal{M}, s \Vdash_\top \psi$ implies $(\forall t \rhd s : \mathcal{M}, t \Vdash_\top \psi$ implies $\mathcal{M}, t \Vdash_\top [\psi]\phi)$
$\iff \mathcal{M}, s \Vdash_\top \psi$ implies $(\forall t \rhd s : \mathcal{M}, t \Vdash_\top \psi$ implies $(\mathcal{M}, t \Vdash_\top \psi$ implies $\mathcal{M}, t \Vdash_{\top\wedge\psi} \phi))$
$\iff \mathcal{M}, s \Vdash_\top \psi$ implies $(\forall t \rhd s : \mathcal{M}, t \Vdash_\top \psi$ implies $\mathcal{M}, t \Vdash_{\top\wedge\psi} \phi)$
$\iff \mathcal{M}, s \Vdash_\top \psi$ implies $(\forall t \rhd s : \mathcal{M}, t \Vdash_\top \top \wedge \psi$ implies $\mathcal{M}, t \Vdash_{\top\wedge\psi} \phi)$
$\iff \mathcal{M}, s \Vdash_\top \psi$ implies $\mathcal{M}, s \Vdash_{\top\wedge\psi} \Box_i\phi$
Thus $\mathcal{M}, s \Vdash [\psi]\Box_i\phi \leftrightarrow (\psi \to \Box_i(\psi \to [\psi]\phi))$ Similarly, we can verify that `!K'` is valid w.r.t. $\Vdash$.

For `EA!`: immediate from the implication form of the truth condition of $[\psi]\phi$. □

Based on the lemmata 2 and 1, we can prove the soundness of **PA** (and some of its extensions) w.r.t. $\Vdash$.

Theorem 3. *For all `PAL` formulas ϕ: $\vdash_{\mathbf{PA}+\texttt{EA!}+\texttt{!K'}} \phi$ implies $\Vdash \phi$.*

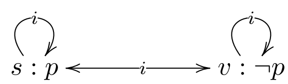
Now we prove that many axioms and rules we mentioned in the introduction are not derivable from **PA**, by showing that they are not valid w.r.t $\Vdash$.

Lemma 3. *None of* !COM,NEC!,RE!,RE *is valid under* $\Vdash$.

PROOF For !COM: We consider $[p][\Box_i p]\bot$ and $[p \wedge [p]\Box_i p]\bot$. From Proposition 10,

$$\Vdash [p][\Box_i p]\bot \leftrightarrow [p \wedge \Box_i p]\bot.$$

Note that $[p]\Box_i p$ is valid w.r.t. $\Vdash$ thus $\Vdash [p]\Box_i p \leftrightarrow \top$. From Proposition 11, $\Vdash [p \wedge [p]\Box_i p]\bot \leftrightarrow [p \wedge \top]\bot \leftrightarrow [p]\bot$. However, $[p \wedge \Box_i p]\bot \leftrightarrow [p]\bot$ is not valid e.g., on the following (S5) model:



For NEC!: It is not hard to verify that $[\neg\Box_i p \vee \neg p](\neg\Box_i p \vee \neg p)$ is valid. From Proposition 10, $\Vdash ([p][\neg\Box_i p \vee \neg p](\neg\Box_i p \vee \neg p)) \leftrightarrow ([p \wedge (\neg\Box_i p \vee \neg p)](\neg\Box_i p \vee \neg p))$. From Proposition 11, $\Vdash ([p][\neg\Box_i p \vee p](\neg\Box_i p \vee \neg p)) \leftrightarrow ([p \wedge \neg\Box_i p](\neg\Box_i p \vee \neg p))$. However, $[p \wedge \neg\Box_i p](\neg\Box_i p \vee \neg p)$ is clearly not valid in the above (S5) model.

For RE! and RE: From the proof of the above case of NEC!, we have a valid equivalence: $([\neg\Box_i p \vee \neg p](\neg\Box_i p \vee \neg p)) \leftrightarrow \top$. However, although $[p]\top$ is still valid, $[p][\neg\Box_i p \vee \neg p](\neg\Box_i p \vee \neg p)$ is not valid, as we have shown. Therefore RE! is not valid w.r.t. $\Vdash$, thus RE is not valid too.

□

From Lemma 3, Theorem 3 we have:

Theorem 4. *None of* !COM, NEC!, RE, RE! *can be derived from* $\mathbf{PA} + \mathtt{EA!} + \mathtt{!K'}$.

PROOF From Theorem 3, for all ϕ: $\nVdash \phi$ implies $\nvdash_{\mathbf{PA}+\mathtt{EA!}+\mathtt{!K'}} \phi$. Moreover, since the rules in $\mathbf{PA} + \mathtt{EA!} + \mathtt{!K'}$ preserve validity, we can show that if a rule is not valid w.r.t. $\Vdash$, then it is not derivable in $\mathbf{PA} + \mathtt{EA!} + \mathtt{!K'}$. However, Lemma 3 says none of !COM, NEC!, RE, RE! is valid w.r.t. $\Vdash$. □

Since DIST! is derivable from $\mathbf{PA} + \mathtt{EA!}$ then the following corollary is immediate:

Corollary 2. $\mathbf{PA} + \mathtt{!K'} + \mathtt{EA!} + \mathtt{DIST!}$ *and its subsystems are not complete w.r.t.* $\vDash$.

We know that $\mathbf{PA}$ + !COM is sound and complete w.r.t. standard semantics. Now we give a complete axiomatization of PAL under $\Vdash$. Recall that !COM$\wedge$ is the axiom schema $[\psi][\chi]\phi \leftrightarrow [\psi \wedge \chi]\phi$. We can show the completeness of **PA**+!COM$\wedge$ w.r.t. our new semantics.

Theorem 5. **PA** + !COM$\wedge$ *is sound and weakly complete w.r.t.* $\Vdash$.

PROOF Soundness follows from Theorem 3 and Proposition 10. For completeness, clearly we can use the reduction axioms in **PA**+!COM$\wedge$ to translate a PAL formula in to an equivalent EL formula w.r.t. $\Vdash$ (cf. the proof of Theorem 1). From proposition 9 and the completeness of **K** w.r.t. $\vDash$, the desired completeness can be obtained. □

Despite the technical motivation behind **PA** + !COM$\wedge$, it also stipulates a particular kind of updates which may be reasonable in modelling real agents. What !COM$\wedge$ says is that the agents are not 'instant updaters' in the sense that they postpone the update until they hear all the consecutive announcements and collect them all together as a conjunction. Here are two realistic scenarios which may exemplify this rationale: 1. two announcements are made right after each other, and in the flash of time between the two agent may not manage to update their information according to the first announcement. Therefore they may take the two announcements as a conjunction; 2. Agents may intentionally postpone the updates according to the announcements: it makes sense if we are considering announcements from different (reliable/unreliable) sources which may contradict each other.

!COM$\wedge$ stipulates a special case of announcement composition different from the standard one. We may well assume that agents have limited memory in remembering the previous announcements and employ different forgetting mechanisms and so on. By stipulating different composition axioms and designing non-standard semantics accordingly, we can model different types of agents/updates. A systematic study of such non-standard PAL is left for future work.

3.2 Another non-standard semantics

The rest of this paper is devoted to the axioms DIST!, SDIST!, !K$'$, EA! and rule NEC!. First note that EA! is valid w.r.t. the above semantics $\Vdash$ thus DIST!, SDIST!, !K$'$ are also valid (by soundness of **PA**$+$EA!). We do not know yet whether these axioms are derivable from **PA**, and moreover it is unclear whether **PA**$+$NEC! is complete. To show that DIST!, SDIST!, !K$'$, and EA! are not derivable in **PA**$+$NEC!, we now define another semantics ($\Vvdash$) which differs from $\vDash$ in the clause of $[\psi]\phi$.

In the sequel, we say that a formula ϕ is *special* if, modulo associativity and commutativity of $\wedge$, $\phi = \bigwedge_{1\leq i\leq n}\phi_i \wedge \bigwedge_{1\leq j\leq m}\phi'_j$ where $n \geq 1$, $m \geq 0$, and ϕ_i are in the shape of $[\chi]\chi'$ but none of ϕ'_j is in such a shape. If ϕ is special then we write $\phi = \phi_{[]} \wedge \phi_{-[]}$ where $\phi_{[]}$ and $\phi_{-[]}$ are the corresponding conjunctions of announcement formulas and non-announcement formulas respectively.

Given a Kripke model over $\mathbf{P}, \mathbf{I}$: $\mathcal{M} = (S, \{\to_i \mid i \in \mathbf{I}\}, V)$, the new truth conditions are as follows:

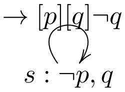
$$\mathcal{M}, s \Vvdash [\psi]\phi \Leftrightarrow \begin{cases} \mathcal{M}, s \Vvdash \phi_{[]} & \text{if } \mathcal{M}, s \not\Vvdash \psi \text{ and } \phi \text{ is special} \\ \mathcal{M}, s \Vvdash \psi \text{ implies } \mathcal{M}|_\psi, s \Vvdash \phi & \text{otherwise} \end{cases}$$

Intuitively, the new semantics for $[\psi]\phi$ depends on the form of ϕ thus RE! is expected to be not valid under this semantics. In the case that ψ is false and ϕ involves announcement formulas, we simply skip the false announcement of ψ (an agent does not go mad if hearing a false announcement followed by other announcements: they can just skip the first one.)

According to this semantics, we can show that DIST!, SDIST!, !K$'$ and EA! are not valid.

Lemma 4. EA!*,* DIST!*,* SDIST!*, and* !K$'$ *are not valid w.r.t.* $\Vvdash$*.*

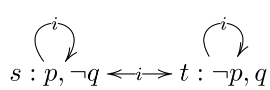
Proof For EA!: Consider $(p \to [p][q]\neg q) \to [p][q]\neg q$ and the following (S5) model $\mathcal{M}$:



It is clear that $\mathcal{M}, s \Vvdash p \to [p][q]\neg q$. However, $\mathcal{M}, s \Vvdash [p][q]\neg q \iff \mathcal{M}, s \Vvdash [q]\neg q \iff (\mathcal{M}, s \Vvdash q$ implies $\mathcal{M}|_q, s \Vvdash \neg q)$. Thus $\mathcal{M}, s \not\Vvdash [p][q]\neg q$.

For DIST! and SDIST!: Consider the above model again, it is easy to verify that $[p](p \to [q]\neg q) \to ([p]p \to [p][q]\neg q)$ is not valid.

For !K$'$: consider $[p]\Box_i[q]\bot \leftrightarrow (p \to \Box_i[p][q]\bot)$ and the following (S5) model:



$\mathcal{M}, s \Vvdash [p]\Box_i[q]\bot \iff \mathcal{M}, s \Vvdash p$ implies $(\mathcal{M}|_p, s \Vvdash \Box_i[q]\bot)$, and $\mathcal{M}, s \Vvdash p \to \Box_i[p][q]\bot \iff (\mathcal{M}, s \Vvdash p$ implies $\mathcal{M}, s \Vvdash \Box_i[p][q]\bot)$. Note that $\mathcal{M}, t \Vvdash [p][q]\bot \iff \mathcal{M}, t \Vvdash [q]\bot \iff \mathcal{M}|_q, t \Vvdash \bot$. Therefore $\mathcal{M}, s \not\Vvdash \Box_i[p][q]\bot$. Thus $\mathcal{M}, s \Vvdash [p]\Box_i[q]\bot$ but $\mathcal{M}, s \not\Vvdash p \to \Box_i[p][q]\bot$. $\Box$

Now we prove that **PA** is sound w.r.t. this semantics.

Compared to $\vDash$, since we do not change the semantics for Boolean formulas and $\Box_i\phi$ formulas, the proof of Lemma 1 also works here w.r.t. $\Vvdash$:

Lemma 5. *TAUT,* MP*,* NECK *and* DISTK *are valid w.r.t.* $\Vvdash$*.*

Lemma 6. !ATOM, !NEG, !CON, *and* !K *are valid w.r.t.* $\Vdash$.

PROOF The case for !ATOM is trivial. !CON is a tricky one and we will see how the complicated case-divided semantics of $[\psi]\phi$ pays back.

For !CON: First note that $\phi \wedge \chi$ is not special iff ϕ and ψ are both not special. Now we consider two cases:

- If $\phi \wedge \chi$ is not special, then neither ϕ nor ψ is special. $\mathcal{M}, s \Vdash [\psi](\phi \wedge \chi) \iff (\mathcal{M}, s \Vdash \psi$ implies $\mathcal{M}|_\psi, s \Vdash \phi \wedge \chi)$
  $\iff \mathcal{M}, s \Vdash \psi$ implies $(\mathcal{M}|_\psi, s \Vdash \phi$ and $\mathcal{M}|_\psi, s \Vdash \chi)$
  $\iff (\mathcal{M}, s \Vdash \psi$ implies $\mathcal{M}|_\psi, s \Vdash \phi)$ and $(\mathcal{M}, s \Vdash \psi$ implies $\mathcal{M}|_\psi, s \Vdash \chi)$
  $\iff (\mathcal{M}, s \Vdash [\psi]\phi$ and $\mathcal{M}, s \Vdash [\psi]\chi)$
- If $\phi \wedge \chi$ is special then at least one of ϕ and χ is special. Suppose w.l.o.g. that χ is not special and ϕ is special thus $\phi = \phi_{[]} \wedge \phi_{-[]}$. Here are again two cases to be considered:
  - suppose $\mathcal{M}, s \Vdash \psi$ then the new semantics coincides with the standard one thus $\mathcal{M}, s \Vdash [\psi](\phi \wedge \chi) \leftrightarrow [\psi]\phi \wedge [\psi]\chi)$.
  - suppose $\mathcal{M}, s \nVdash \psi$ then $\mathcal{M}, s \Vdash [\psi](\phi \wedge \chi) \iff \mathcal{M}, s \Vdash \phi_{[]}$
    $\iff \mathcal{M}, s \Vdash \phi_{[]}$ and $(\mathcal{M}, s \Vdash \psi \implies \mathcal{M}|_\psi, s \Vdash \chi)$
    $\iff \mathcal{M}, s \Vdash [\psi]\phi$ and $\mathcal{M}, s \Vdash [\psi]\chi$

The proofs for !NEG and !K are almost as before under the standard semantics $\vDash$. We only need to handle the extra special cases. Now suppose ϕ is special. Clearly $\neg\phi$ and $\Box_i\phi$ are not special.

For !NEG: We only need to consider the case when $\mathcal{M}, s \nVdash \psi$ since otherwise the proof for the standard semantics suffices. Then it is clear that $\mathcal{M}, s \Vdash \psi \to \neg[\psi]\phi$ and $\mathcal{M}, s \Vdash [\psi]\neg\phi$ since $\neg\phi$ is not special. Thus $\mathcal{M}, s \Vdash [\psi]\neg\phi \leftrightarrow (\psi \to \neg[\psi]\phi)$.

For !K: It is clear that if $\mathcal{M}, s \nVdash \psi$ then $\mathcal{M}, s \Vdash [\psi]\Box_i\phi \leftrightarrow (\psi \to (\Box_i(\psi \to [\psi]\phi)))$. However, it does not suffice since even ψ is true at $\mathcal{M}, s$ it is still possible that $\psi \to (\Box_i(\psi \to [\psi]\phi))$ differs from the standard semantics due to the appearance of $[\psi]\phi$ in the scope of $\Box_i$. Now suppose $\mathcal{M}, s \Vdash \psi$. $\mathcal{M}, s \Vdash [\psi]\Box_i\phi \iff (\mathcal{M}, s \Vdash \psi$ implies $\mathcal{M}|_\psi, s \Vdash \Box_i\phi) \iff \mathcal{M}|_\psi, s \Vdash \Box_i\phi$. On the other hand, $\mathcal{M}, s \Vdash \psi \to \Box_i(\psi \to [\psi]\phi) \iff (\mathcal{M}, s \Vdash \psi$ implies $\mathcal{M}, s \Vdash \Box_i(\psi \to [\psi]\phi)) \iff \mathcal{M}, s \Vdash \Box_i(\psi \to [\psi]\phi) \iff (\forall t \rhd s : \mathcal{M}, t \Vdash \psi$ implies $(\mathcal{M}, t \Vdash [\psi]\phi)$. Note that the new semantics only differs from the standard one if ψ is false. Thus $(\forall t \rhd s : \mathcal{M}, t \Vdash \psi$ implies $(\mathcal{M}, t \Vdash [\psi]\phi) \iff (\forall t \rhd s : \mathcal{M}, t \Vdash \psi$ implies $(\mathcal{M}, t \Vdash \psi$ implies $\mathcal{M}|_\psi \Vdash \phi) \iff (\forall t \rhd s : \mathcal{M}, t \Vdash \psi$ implies $\mathcal{M}|_\psi, t \Vdash \phi) \iff (\forall t \rhd s : t$ exists in $\mathcal{M}|_\psi$ implies $\mathcal{M}|_\psi, t \Vdash \phi) \iff \mathcal{M}|_\psi, s \Vdash \Box_i\phi$.

□

Moreover, we can show that NEC! is valid w.r.t. $\Vdash$.

Lemma 7. NEC! *is valid under* $\Vdash$.

PROOF Suppose $\Vdash \phi$. Now consider $[\psi]\phi$. There are two cases:

- ϕ is not special: Trivial.
- ϕ is special: It is in the shape of $\phi_{[]} \wedge \phi_{-[]}$. To verify $\mathcal{M}, s \Vdash [\psi]\phi$ there are again two cases. Suppose $\mathcal{M}, s \Vdash \psi$, then $\mathcal{M}, s \Vdash [\psi]\phi \iff \mathcal{M}|_\psi, s \Vdash \phi$ which is true since $\Vdash \phi$. Now suppose $\mathcal{M}, s \nVdash \psi$, then $\mathcal{M}, s \Vdash [\psi]\phi \iff \mathcal{M}, s \Vdash \phi_{[]}$. Since $\Vdash \phi$, $\Vdash \phi_{[]} \wedge \phi_{-[]}$ thus $\Vdash \phi_{[]}$. Therefore, $\mathcal{M}, s \Vdash \phi_{[]}$. This concludes the proof.

□

Lemmata 5, 6, and 7 showed that **PA** + NEC! is sound w.r.t. $\Vdash$. Together with 4 we have:

Theorem 6. *None of* DIST!, SDIST! !K′, EA! *can be derived from* **PA** + NEC!.

As an immediate corollary:

Corollary 3. **PA** + NEC! *is not complete w.r.t. standard semantics* $\vDash$.

4 Discussion and future work

We have shown that **PA** and many natural extensions of it are not complete w.r.t. the standard semantics on arbitrary Kripke models. It is also natural to ask whether **PA** plus the usual S5 axioms: T $(\phi \to \Diamond\phi)$, 4:$(\Box\phi \to \Box\Box\phi)$, B: $(\phi \to \Box\Diamond\phi)$ is complete under standard semantics on S5 models. We conjecture that all of our incompleteness results still hold if we replace **PA** by $\mathbf{PA} + \mathtt{T} + \mathtt{4} + \mathtt{B}$. For now, to be more confident about this conjecture, observe that we only use S5 counter models to show the invalidity of axioms and rules.

Similar discussions can be carried out in the context of PAL with common knowledge and dynamic epistemic logic with action models (cf. e.g., [1,9]). We hope that this paper has demonstrated the subtleness of the axiomatizations of dynamic epistemic logics and the importance and the use of the composition axiom in various forms.

As we mentioned, the composition axioms are not just technical patches to make the system complete, actually they are essential components stating how the updates are interacting with each other, thus deserving independent study. To accompany different composition axioms we may design different non-standard semantics and keep the expressive power the same. One possible application is to use reasonable composition axioms (e.g., !COM$\wedge$) to control the iteration of updates which is responsible for the undecidability in the standard setting (cf. [7]). Different composition axioms may also be useful in the setting of iterated belief revision (cf. e.g., [3]). The syntactic driven approaches to dynamic epistemic logic will bring new insights and raise new questions to this flourishing field of research.

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