Not all those who wander are lost: dynamic epistemic reasoning in navigation

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Abstract

In everyday life, people get lost even when they have the map: they simply may not know where they are in the map. However, when moving forward they may have new observations which can help to locate themselves by reasoning. In this paper, we propose and develop a semantic-driven dynamic epistemic framework to handle epistemic reasoning in such navigation scenarios. Our framework can be viewed as a careful blend of dynamic epistemic logic and epistemic temporal logic, thus enjoying features from both frameworks. We made an in-depth study on many model theoretical aspects of the proposed framework and provide a complete axiomatization.

Keywords: dynamic epistemic logic, epistemic temporal logic, navigation, planning

1 Introduction

1.1 Motivation

Have you ever been lost with a map? Almost everyone had such an experience as a tourist in an unfamiliar city: even when you have the map of the city, it is sometimes still hard to figure out where you are exactly and how to reach your destination. There are typical cases when you just cannot find the street name, or you are on a long long street with a lot of turns (welcome to Amsterdam!). In such scenarios, a little bit of wandering and reasoning may help:

> *This circular street along the canal is called Prinsengracht, but I am not sure whether I am at place A or B. Let me walk a bit further. Now I see that I can turn left but according to the map if I were at B I would not be able to turn left so soon. Thus I must have been A. Now I know my way to Leidseplein.*

In the above reasoning process, the important elements are: the map, the uncertainties about your location and your observations of the current available

[1] The main title is taken from the poem *All that is gold does not glitter* by J.R.R. Tolkien.

[2] This version differs from the published version in the naming of one of the axioms. Instead of the improper name *no learning*, we call it *observational no miracle* in this version.

actions. We reason by matching the actual available moves with the moves at the possible current locations according to the map.

Sometimes, it is more important to reach your destination than locating yourself exactly. In the *Mission Impossible*-like films, the secret agent sneaking in an enemy building is usually guided by his headquarters (often a geek sitting behind a laptop). However, the communication with the HQ will almost always be lost at some point for some reason. Finally the agent has to find his own way. Suppose the agent has the following floor plan with safety zones marked (though there are no special signs at those places), but does not know whether he is currently at s_2 or at s_3 (denoted by the dashed line):

$$\begin{tikzcd}
s_7 & s_6 \arrow[l, "l"'] & s_8 : \mathsf{Safe} & s_9 : \mathsf{Safe} & \\
s_1 \arrow[r, "r"] & s_2 \arrow[u, "u"] \arrow[r, "r"] \arrow[r, dash, dotted, bend left] & \underline{s_3} \arrow[u, "u"] \arrow[r, "r"] & s_4 : \mathsf{Safe} \arrow[u, "u"] \arrow[r, "r"] & s_5
\end{tikzcd}$$

Now suppose that the agent is *actually* at s_3 (the underlined state) and he can *only* observe the available moves at his current location, e.g., at s_3 he only observes that he may move right (r) or move up (u). Let us consider the following scenarios:

- Knowing the actual location of the agent, the HQ may guide the agent to move right (do r) to a safe place (s_4). However, merely following the command, the agent may not know that he is safe after doing r, since if he were at s_2, doing r would get him to a unsafe place s_3, but s_3 and s_4 share exactly the same available moves (r and u), thus he cannot distinguish them.
- The HQ may alternatively guide the agent to move up (u) to s_8. This time the agent should know that he is safe: he sees that he cannot move any further, however, if he were at s_2 initially and thus at s_6 after moving u, then he would be able to move left which contradicts his current observations.
- Suppose the communication with the HQ is lost, the agent may make his own plan as follows: he knows that no matter where exactly he is right now, moving first r and then u will make sure that he is safe, although afterwards he will still not know where he is exactly.

In this paper, we formalize the epistemic reasoning behind such scenarios by proposing a semantic-driven dynamic epistemic logical framework with the following real life applications in mind as the long term goals:

- Global navigation: given the map with uncertainties and the actual location of a subject (human or robot), navigate it to guarantee certain (epistemic) goals, e.g., transport the prisoners to a safe place without letting them know where they are.
- Local navigation: given only the map with uncertainties of the current location of a subject, let the subject navigate itself to guarantee certain (epistemic) goals, e.g., the robot should plan its own way in an endangered nuclear power plant to make sure it “knows” that it will reach all the critical

machines that need to be shut down.

1.2 Related work

Related Work *Dynamic epistemic logics* (DEL) are designed to handle knowledge updates caused by events (cf. e.g, [20,1,24]). Arguably the most general framework of DEL is the one using event models proposed in [2]. It is natural to apply the existing techniques of DEL with event models in the navigation setting which is also about knowledge updates after actions. However, as we will show in the later part of the paper, the standard event model approach (even extended with protocols as in [21,11]) is not suitable to handle epistemic reasoning in such scenarios. On the other hand, algebraic approaches inspired by DEL have been proposed to model the robot navigation in [17,18,10]. Despite the apparent differences in frameworks (algebra vs. logic), we depart from this series of works in the way of handling the map information and actions. In [17,18,10], the nodes of the map are encoded by basic propositions and thus the moves in the map are taken to be actions that change the truth value of basic propositions (encodings of the current position). In our semantic-driven approach, we simply take the maps with uncertainties as models and moving in a map does not change the truth values of any basic propositions but the current position and epistemic uncertainties. Instead of the *theorem proving* in the algebraic approach we can *model check* a rich class of desired properties expressed by a natural yet simple logic language, which can be fully automated.

Another usual framework for reasoning about knowledge and developments of a system is the *epistemic temporal logic* (ETL) proposed in [7,19]. Essentially, ETL and DEL are instances of two-dimensional modal logic, which is also used in other multi-agent systems (cf. e.g. [16,14]). Efforts have been made to merge the frameworks of ETL and DEL [21,11]. Our approach can also be viewed as a careful blend of ETL and DEL in the sense that the temporal development is explicitly encoded in the map as in ETL but the epistemic developments are computed in spirit of DEL.

The planning problem with uncertainties and non-deterministic actions (conformant planning) are well-studied in Artificial Intelligence (cf. e.g., [8]), since it was raised in [15]. Our models are similar to the belief spaces used in solving such planning problems (cf. e.g., [4]). The focus there, however, is on the algorithms and heuristics to the planning problem while we would like to present a semantics-driven logic for reasoning about knowledge, which also differs from the situation calculus based logical planning approaches such as [13]. We hope to encode various planning problems by model checking problems in the extensions of our framework, which we leave for further occasions. [3]

[3] The connections to belief space planning was suggested to us by Prof. Bernhard Nebel, Dr. Christian Becker-Asano and Dr. Andreas Witzel, when the first author was visiting Isaac Newton Institute in 2012 for a project coordinated by Prof. Benedikt Löwe.

The technical contributions and the structure of the paper are summarized as follows:

- In Section 2, we propose a dynamic epistemic framework on maps with uncertainties. The semantics is non-standard in the sense that we only assign truth values to the formulas on certain states of the models (not all of them!).
- A substitution-closed axiomatization is provided in Section 3 to capture the validity of the logic and the completeness is proved by using a detour technique handling the interactions of the epistemic operator and the action operators.
- Section 4 discusses some model theoretical properties of the proposed logic: the structural invariance, the finite model property and notably a non-trivial normal form theorem which says that any formula is equivalent to an (exponentially longer) formula where K operator only appear outside the scopes of action operators.
- In Section 5, we compare our logic with ETL via an intuitive translation. We also show that, due to technical reasons, DEL with event models and protocols are not suitable for handling navigation tasks compared to our logic.

2 Preliminaries

2.1 Kripke model with uncertainties

Given a set **P** of basic propositions and a set **A** of basic actions, a *multimodal Kripke model* $\mathcal{N}$ w.r.t. **P** and **A** is a tuple: $\mathcal{N} = \langle S, \{R_a \mid a \in \mathbf{A}\}, V\rangle$ where S is a non-empty set of states (or locations), $R_a \subseteq S \times S$ is a binary relation, $V : \mathbf{P} \to \mathcal{P}(S)$ is a valuation function. To simplify notations, we write $s \xrightarrow{a} t$ for sR_at. Given a Kripke model $\mathcal{N}$, we denote its set of states, relations and valuation by $S_\mathcal{N}$, $\xrightarrow{a}_\mathcal{N}$ and $V_\mathcal{N}$. Given an $s \in S_\mathcal{N}$, let $e(s)$ be the set of available actions at s, i.e., $e(s) = \{a \mid \exists s' \in S_\mathcal{N} \text{ such that } s \xrightarrow{a} s'\}$. Such a Kripke model may be viewed as an abstract "map" with some basic facts decorating the states. Note that *non-deterministic* actions are allowed: executing a at the same state may result in different states. [4]

An *uncertainty map (UM)* is a Kripke model with a set of uncertainties about the current location of an agent. Formally, a UM model $\mathcal{M}$ is a tuple

$$\langle S, \{R_a \mid a \in \mathbf{A}\}, V, U\rangle$$

where $\langle S, \{R_a \mid a \in \mathbf{A}\}, V\rangle$ is a Kripke model and $U \subseteq S$ is a non-empty set such that for all $s, t \in U$: $e(s) = e(t)$. The requirement of U actually says that the uncertainties should comply with the observation about the available actions. We use $U_\mathcal{M}$ to denote the uncertainty set of $\mathcal{M}$. A *pointed UM model*

[4] The non-determinism can also be used to model uncertainties about actions, e.g., without a compass, moving east, west, south, north may look exactly the same to an agent at a cross road, thus in the map we may use the same action to stand for these four moves.

$(\mathcal{M}, s)$ is a UM model $\mathcal{M}$ with a designated state $s \in U_{\mathcal{M}}$ representing the actual location of the agent. Given a model $\mathcal{M}$, let $E(s)$ be the set of states that share the same available actions as s, i.e., $E(s) = \{t \in S \mid e(s) = e(t)\}$.

The graph mentioned in the introduction can be viewed as an illustration of a UM model w.r.t $\mathbf{P} = \{\text{Safe}\}$[5] and $\mathbf{A} = \{l, u, r\}$ with the uncertainty set $\{s_2, s_3\}$ (the states connected by the dotted line).

2.2 Language and semantics

To reason about knowledge and actions in the scenarios mentioned earlier, we use the following simplest modal language $\mathtt{EAL}^{\mathbf{A}}_{\mathbf{P}}$ (*Epistemic Action Language*) with knowledge and actions as modalities:

$$\phi ::= \top \mid p \mid \neg\phi \mid \phi \wedge \phi \mid [a]\phi \mid K\phi$$

where $p \in \mathbf{P}$, $a \in \mathbf{A}$. As usual, we use the following abbreviations: $\bot := \neg\top$, $\phi \vee \psi := \neg(\neg\phi \wedge \neg\psi), \phi \to \psi := \neg\phi \vee \psi, \langle a\rangle\phi := \neg[a]\neg\phi, \hat{K}\phi := \neg K\neg\phi$. Intuitively, $K\phi$ says that the agent knows that ϕ and $[a]\phi$ expresses that if the agent can move forward by a, then after doing a, ϕ holds (a may be non-deterministic).

Given any UM model $\mathcal{M} = \langle S, \{R_a \mid a \in \mathbf{A}\}, V, U\rangle$ and any point $s \in U$ the satisfaction relation is defined on pointed UM model $\mathcal{M}, s$ as:

$$\begin{array}{rcl}
\mathcal{M}, s \vDash \top & \iff & \text{always} \\
\mathcal{M}, s \vDash p & \iff & s \in V(p) \\
\mathcal{M}, s \vDash \neg\phi & \iff & \mathcal{M}, s \nvDash \phi \\
\mathcal{M}, s \vDash \phi \wedge \psi & \iff & \mathcal{M}, s \vDash \phi \text{ and } \mathcal{M}, s \vDash \phi \\
\mathcal{M}, s \vDash [a]\phi & \iff & \forall t \in S : s \xrightarrow{a} t \text{ implies } \mathcal{M}|^a_t, t \vDash \phi \\
\mathcal{M}, s \vDash K\phi & \iff & \forall u \in U : \mathcal{M}, u \vDash \phi
\end{array}$$

where $\mathcal{M}|^a_t = \langle S, \{R_a \mid a \in \mathbf{A}\}, V, U|^a_t\rangle$ and $U|^a_t = U|^a \cap E(t)$ with $U|^a = \{r' \mid \exists r \in U \text{ such that } r \xrightarrow{a} r'\}$.

It is easy to check that in the clause of $[a]\phi$, $U|^a_t \subseteq E(t)$ and $t \in U|^a_t$ thus $\mathcal{M}|^a_t, t$ is indeed a pointed UM model.

The semantics of $K\phi$ is rather intuitive as in epistemic logic. The intuition behind the semantics of $[a]\phi$ formulas is as follows: if you can move forward by a and then end up at t, your uncertainty set should be *carried forward* with you along the possible a moves, which explains the first set in the definition of $U|^a_t$; As for the second part, note that you may eliminate some uncertainties according to the actual observation about the available actions at t.

Here are a few points we have to highlight before moving further:

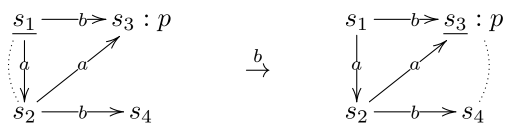
- We define semantics on pointed UM models and *only* the states in $U_{\mathcal{M}}$ can be taken as the designated points to evaluate formulas. This means that the truth values of $\mathtt{EAL}^{\mathbf{A}}_{\mathbf{P}}$ formulas are *not* defined on all the states in a model.

[5] If there is no label *Safe* at a state, then it means the proposition *Safe* is not true there.

- In particular, your knowledge at a certain state in the model *only* become clear when you have moved there or one of its indistinguishable states, thus the knowledge is essentially path-dependent (see the example below).
- Therefore, we say that a formula ϕ is *valid* ($\vDash \phi$) iff for any pointed UM model $\mathcal{M}, s$: $\mathcal{M}, s \vDash \phi$.

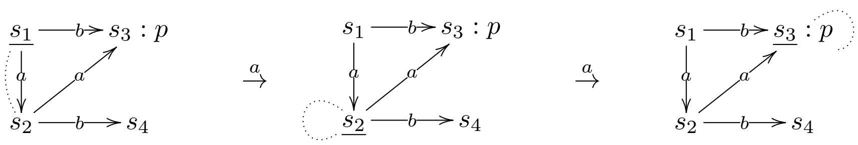
Let us consider the following example (it is a tweaked version of a common example used in [17,18,10]).

Example 2.1 The left and right graphs below depict the initial pointed model $\mathcal{M}, s_1$, and the pointed model after a b move $(\mathcal{M}|^b_{s_3}, s_3)$ respectively.



It is easy to verify that $\mathcal{M}, s_1 \vDash K\neg p \wedge \langle b\rangle\neg Kp$.

The left, middle, and right graphs below depict the pointed models $\mathcal{M}, s_1$, $\mathcal{M}|^a_{s_2}, s_2$ and $(\mathcal{M}|^a_{s_2})|^a_{s_3}, s_3$ respectively.



Now we see that $\mathcal{M}, s_1 \vDash K\neg p \wedge \langle a\rangle\langle a\rangle Kp$. Compare $(\mathcal{M}|^a_{s_2})|^a_{s_3}, s_3$ and $\mathcal{M}|^b_{s_3}, s_3$, it is clear that checking whether Kp is true at s_3 depends on how do you get to s_3. It does not mean much to evaluate the knowledge of an agent on the states that he thinks he cannot be currently. The agent may know more or stay ignorant after wandering around.

Going back to our "mission impossible" example in the introduction, we can now verify the claims about three scenarios w.r.t. the model (call it $\mathcal{M}_{\mathrm{MI}}$):

- $\mathcal{M}_{\mathrm{MI}}, s_3 \vDash \langle r\rangle(\text{Safe} \wedge \neg K\text{Safe})$ (HQ guides you safe but you do not know it)
- $\mathcal{M}_{\mathrm{MI}}, s_3 \vDash \langle u\rangle(\text{Safe} \wedge K\text{Safe})$ (HQ guides you safe and you know it)
- $\mathcal{M}_{\mathrm{MI}}, s_3 \vDash K(\langle r\rangle\langle u\rangle\text{Safe} \wedge [r][u]\text{Safe})$ (You know the plan will make you safe)

Given a UM model pointed $\mathcal{M}, s$, a goal expressed by a $\mathtt{EAL}^{\mathbf{A}}_{\mathbf{P}}$ formula ϕ and a plan as a sequence of actions $a_1 \cdots a_n$, we can verify whether the plan can possibly satisfy the goal by checking $\mathcal{M}, s \vDash \langle a_1\rangle \cdots \langle a_n\rangle\phi$.

3 Axiomatization

In this section, we provide a sound and complete axiomatization of $\mathtt{EAL}^{\mathbf{A}}_{\mathbf{P}}$ on UM models. Recall that a formula is valid if it holds on all the *pointed models*. In the sequel we assume that $\mathbf{A}$ is finite.

Given a UM model $\mathcal{M}$, let $\mathcal{M}^{\mathtt{ML}}$ be the Kripke "core" of $\mathcal{M}$ (by simply ignoring the uncertainty set $U_{\mathcal{M}}$); let $\mathcal{M}^{\mathtt{EL}}$ be the **S5** model $\langle U_{\mathcal{M}}, \sim, V' \rangle$ where $\sim\, = U_{\mathcal{M}} \times U_{\mathcal{M}}$ and $V' = V_{\mathcal{M}}|_{U_{\mathcal{M}}}$. Let $\vDash_{\mathtt{ML}}$ and $\vDash_{\mathtt{EL}}$ denote the standard semantics for multimodal logic and epistemic logic respectively (cf. e.g., [3]).

Two easy observations follow immediately from the semantics of $\mathtt{EAL}^{\mathbf{A}}_{\mathbf{P}}$:

Proposition 3.1 *For any K-free $\mathtt{EAL}^{\mathbf{A}}_{\mathbf{P}}$-formula ϕ: $\mathcal{M}, s \vDash \phi$ iff $\mathcal{M}^{ML}, s \vDash_{ML} \phi$. For any $[\cdot]$-free $\mathtt{EAL}^{\mathbf{A}}_{\mathbf{P}}$-formula ϕ: $\mathcal{M}, s \vDash \phi$ iff $\mathcal{M}^{EL}, s \vDash_{EL} \phi$.*

However, it is clear that $\mathtt{EAL}^{\mathbf{A}}_{\mathbf{P}}$ cannot be reduced, qua expressive power, to any of these two fragments of $\mathtt{EAL}^{\mathbf{A}}_{\mathbf{P}}$, due to the two dimensional nature (action and knowledge) of the UM models. [6] This means that the usual axiomatization of DEL-style logic (e.g., [20] and [1]) via reductions does not work here. In the axiomatization, we include the axioms of epistemic logic and multimodal logic with extra axioms capturing the dynamics in terms of the interaction between $\langle a \rangle$ and K. Inspired by [25], the Henkin-style completeness proof makes use of an auxiliary semantics which transforms dynamics of models into static relations in the canonical model.

3.1 Finite axiomatization $\mathbf{S}_{\mathtt{EAL}^{\mathbf{A}}_{\mathbf{P}}}$

System $\mathbf{S}_{\mathtt{EAL}^{\mathbf{A}}_{\mathbf{P}}}$

Axioms		**Rules**	
TAUT	all the axioms of propositional logic	MP	$\dfrac{\phi, \phi \to \psi}{\psi}$
DISTK	$K(p \to q) \to (Kp \to Kq)$	NECK	$\dfrac{\phi}{K\phi}$
DIST(a)	$[a](p \to q) \to ([a]p \to [a]q)$	NEC(a)	$\dfrac{\phi}{[a]\phi}$
OBS(a)	$K\langle a \rangle \top \vee K\neg\langle a \rangle \top$	SUB	$\dfrac{\phi(p)}{\phi(\psi)}$
T	$Kp \to p$		
4	$Kp \to KKp$		
5	$\neg Kp \to K\neg Kp$		
PR(a)	$\langle a \rangle \hat{K} p \to \hat{K} \langle a \rangle p$		
ONM(a)	$\bigwedge_{\mathbf{B} \subseteq \mathbf{A}} (\hat{K}\langle a \rangle (p \wedge \psi_{\mathbf{B}}) \to [a](\psi_{\mathbf{B}} \to \hat{K} p))$		

where a ranges over $\mathbf{A}$, p, q range over $\mathbf{P}$ and in the last clause, $\psi_{\mathbf{B}} = (\bigwedge_{b \in \mathbf{B}} \langle b \rangle \top) \wedge (\bigwedge_{b \notin \mathbf{B}} \neg \langle b \rangle \top)$. Since $\mathbf{A}$ is finite, $\mathtt{EAL}^{\mathbf{A}}_{\mathbf{P}}$ is a finite axiomatic system. PR($\cdot$) denotes the axiom of *perfect recall* following the convention in the literature (cf. [9]). ONM($\cdot$) denotes the axiom of *observational no miracle*. It has roughly the structure of the *no miracle* axiom $\hat{K}\langle a \rangle p \to [a]\hat{K} p$, which differs from the well-known *no learning* axiom $\hat{K}\langle a \rangle p \to \langle a \rangle \hat{K} p$ in the modality of a

[6] E.g., you cannot find an epistemic formula to tell two UM models apart if they share the same epistemic core but are quite different in the map part. Similar for the K-free formulas.

in the consequent. Note that exactly this difference decides whether the agent can learn from moving forward (cf.[25] for further discussions). We shall come back to the difference between $\texttt{ONM}(\cdot)$ and the usual no miracle axiom from the semantic point of view in Section 5.

Based on the semantics of $\texttt{EAL}^{\mathbf{A}}_{\mathbf{P}}$ and Proposition 3.1, it is easy to verify that the following axioms and rules are valid: $\texttt{DISTK}, \texttt{DIST}(\cdot), \texttt{T}, 4, 5, \texttt{NECK}, \texttt{NEC}(\cdot)$. The validity of $\texttt{OBS}(\cdot)$ is due to the requirement on the uncertainty sets in UM models. Note that the uniform substitution SUB is also valid according to our semantics, which is different from the usual DEL-style logics (cf. [24]).

To prove the soundness of $\texttt{EAL}^{\mathbf{A}}_{\mathbf{P}}$, we still need to show that $\texttt{PR}(\cdot)$ and $\texttt{ONM}(\cdot)$ are valid. In the following, we verify the corresponding axiom schemas where p is replaced by an arbitrary ϕ.

Proposition 3.2 *For any $a \in \boldsymbol{A}$: $\vDash \langle a\rangle \hat{K}\phi \to \hat{K}\langle a\rangle\phi$*

Proof For any $\mathcal{M}, s$, if $\mathcal{M}, s \vDash \langle a\rangle\hat{K}\phi$, then there is a $t \in S$, such that $s \xrightarrow{a} t$ and $\mathcal{M}|^a_t, t \vDash \hat{K}\phi$, thus there is also a $v \in U_{\mathcal{M}}|^a_t$, $\mathcal{M}|^a_t, v \vDash \phi$. Because $v \in U_{\mathcal{M}}|^a_t = U_{\mathcal{M}}|^a \cap E(t)$, then there is a $u \in U_{\mathcal{M}}$, such that $u \xrightarrow{a} v$ and $E(v) = E(t)$. Thus $U_{\mathcal{M}}|^a_v = U_{\mathcal{M}}|^a_t$, then $\mathcal{M}|^a_t = \mathcal{M}|^a_v$, thus $\mathcal{M}|^a_v, v \vDash \phi$. Since $u \xrightarrow{a} v$, $\mathcal{M}, u \vDash \langle a\rangle\phi$, because $u \in U_{\mathcal{M}}$ then $\mathcal{M}, s \vDash \hat{K}\langle a\rangle\phi$. □

Proposition 3.3 *For any $a \in \boldsymbol{A}$: $\vDash \bigwedge_{\boldsymbol{B}\subseteq\boldsymbol{A}}(\hat{K}\langle a\rangle(\phi \wedge \psi_{\boldsymbol{B}}) \to [a](\psi_{\boldsymbol{B}} \to \hat{K}\phi))$*

Proof For any $\mathcal{M}, s$, we need to prove that for any $\mathbf{B} \subseteq \mathbf{A}$, $\mathcal{M}, s \vDash \hat{K}\langle a\rangle(\phi \wedge \psi_{\mathbf{B}}) \to [a](\psi_{\mathbf{B}} \to \hat{K}\phi)$. If $\mathcal{M}, s \vDash \hat{K}\langle a\rangle(\phi \wedge \psi_{\mathbf{B}})$, then there is a $u \in U_{\mathcal{M}}$, such that $\mathcal{M}, u \vDash \langle a\rangle(\phi \wedge \psi_{\mathbf{B}})$, thus there is also a $v \in S_{\mathcal{M}}$, such that $u \xrightarrow{a} v$ and $\mathcal{M}|^a_v, v \vDash \phi \wedge \psi_{\mathbf{B}}$. Then we need to prove that $\mathcal{M}, s \vDash [a](\psi_{\mathbf{B}} \to \hat{K}\phi)$. Namely, for any $t \in S_{\mathcal{M}}$, assuming $s \xrightarrow{a} t$ and $\mathcal{M}|^a_t, t \vDash \psi_{\mathbf{B}}$, we need to show that $\mathcal{M}|^a_t, t \vDash \hat{K}\phi$. Since $\mathcal{M}|^a_v, v \vDash \psi_{\mathbf{B}}$, then $E(t) = E(v)$, thus $U_{\mathcal{M}}|^a_t = U_{\mathcal{M}}|^a_v$ and $\mathcal{M}|^a_t = \mathcal{M}|^a_v$. Since $\mathcal{M}|^a_v, v \vDash \phi$, $\mathcal{M}|^a_t, v \vDash \phi$. Now since $v \in U_{\mathcal{M}}|^a_t$, $\mathcal{M}|^a_t, t \vDash \hat{K}\phi$. □

Since we include $\texttt{DIST}(\cdot), \texttt{DISTK}, \texttt{NEC}(\cdot), \texttt{NECK}$ in the system, it is easy to verify the following propositions as standard exercises in *normal modal logic*.

Proposition 3.4 $\vdash [a](\phi \wedge \psi) \leftrightarrow ([a]\phi \wedge [a]\psi)$, $\vdash [a]\phi \vee [a]\psi \to [a](\phi \vee \psi)$, $\vdash K(\phi \wedge \psi) \leftrightarrow (K\phi \wedge K\psi)$.

Proposition 3.5 *If $\vdash \phi \leftrightarrow \phi'$, $\vdash \psi \leftrightarrow \psi'$, then $\vdash \neg\phi \leftrightarrow \neg\phi'$, $\vdash \phi \wedge \psi \leftrightarrow \phi' \wedge \psi'$, $\vdash \langle a\rangle\phi \leftrightarrow \langle a\rangle\phi'$, and $\vdash K\phi \leftrightarrow K\phi'$.*

Based on the above propositions, we can show the useful inference rule of *replacements of equivalents* is an admissible rule of the system $\mathbf{S}_{\texttt{EAL}^{\mathbf{A}}_{\mathbf{P}}}$.

Proposition 3.6 *If $\vdash \psi \leftrightarrow \psi'$, and ϕ' is obtained by replacing some occurrences of ψ in ϕ with ψ', then $\vdash \phi \leftrightarrow \phi'$.*

3.2 Completeness

To prove the completeness, we will use an auxiliary semantics of $\texttt{EAL}^{\mathbf{A}}_{\mathbf{P}}$ on *epistemic multimodal models (EM models)*. Formally, an EM model $\mathcal{N}$ is a tuple $\langle S, \{R_a \mid a \in \mathbf{A}\}, V, \sim\rangle$, where $\langle S, \{R_a \mid a \in \mathbf{A}\}, V\rangle$ is a multimodal Kripke model and $\sim$ is an equivalence relation over S such that $s \sim t$ implies $e(s) = e(t)$. Note that $\sim$ can also be viewed as a partition of S. Therefore the difference between a UM model $\mathcal{M}$ and an EM model $\mathcal{N}$ is that $U_{\mathcal{M}}$ denotes a single equivalence class while $\sim$ denotes a set of the equivalence classes which form a partition of $S_{\mathcal{N}}$. $\mathtt{EAL}^{\mathtt{A}}_{\mathtt{P}}$ formulas can be interpreted on EM models with the usual Kripke semantics (denoted as $\Vdash$):

$$\mathcal{N}, s \Vdash [a]\phi \iff \forall t : s \xrightarrow{a} t \text{ implies } \mathcal{N}, t \Vdash \phi$$

$$\mathcal{N}, s \Vdash K\phi \iff \forall t : s \sim t \text{ implies } \mathcal{N}, t \Vdash \phi$$

However, we cannot always transform a UM model $\mathcal{M}$ into an EM model $\mathcal{M}'$ by simply adding more equivalence classes such that for any $\mathtt{EAL}^{\mathtt{A}}_{\mathtt{P}}$-formula ϕ: $\mathcal{M}, s \vDash \phi \iff \mathcal{M}', s \Vdash \phi$. In Example 2.1, based on the initial UM model, it is impossible to assign an equivalence class including s_3 to make sure that $\langle b\rangle\neg Kp$ and $\langle a\rangle\langle a\rangle Kp$ both hold at s_1.

On the other hand, an EM model can also be viewed as a UM model with extra epistemic information. Given an EM model $\mathcal{N} = \langle S, \{R_a \mid a \in \mathbf{A}\}, V, \sim\rangle$ and $s \in S$, let $\mathcal{N}_s$ be the UM model $\langle S, \{R_a \mid a \in \mathbf{A}\}, V, U_s\rangle$ where $U_s = \{t \mid s \sim t \text{ in } \mathcal{N}\}$. We say $\vDash$ and $\Vdash$ *coincide* on an EM model $\mathcal{N}$ if for any $s \in S_{\mathcal{N}}$ and any $\mathtt{EAL}^{\mathtt{A}}_{\mathtt{P}}$-formula ϕ: $\mathcal{N}, s \Vdash \phi \iff \mathcal{N}_s, s \vDash \phi$. It is not hard to see that the two semantics do not coincide on arbitrary EM models in general, however, as we will show later, the two semantics do coincide on the canonical EM model which is essential in the proof of completeness.

Our proof strategy can be summarised as follows:

(i) Prove the Lindebaum-like lemma: every $\mathbf{S}_{\mathtt{EAL}^{\mathtt{A}}_{\mathtt{P}}}$-consistent set of formulas can be extended into a maximal consistent set ($\mathbf{S}_{\mathtt{EAL}^{\mathtt{A}}_{\mathtt{P}}}$-MCS).

(ii) Construct a *canonical EM model* $\mathcal{C}$ and prove the truth lemma w.r.t. the auxiliary semantics ($\Vdash$).

(iii) Show that $\vDash$ and $\Vdash$ coincide on the canonical model thus obtaining the truth lemma w.r.t. $\vDash$. Finally, given a $\mathbf{S}_{\mathtt{EAL}^{\mathtt{A}}_{\mathtt{P}}}$-MCS Γ, $(\mathcal{C}_\Gamma, \Gamma)$ is the pointed UM model which satisfies all the formulas in Γ.

The Lindebaum lemma is routine. We define a canonical EM model based on MCSs of $\mathbf{S}_{\mathtt{EAL}^{\mathtt{A}}_{\mathtt{P}}}$ as usual for normal modal logics (cf. e.g., [3]):

$$\mathcal{C} = < S^c, \{R^c_a \mid a \in \mathbf{A}\}, \sim^c, V^c >$$

where:

- S^c is the set of all $\mathbf{S}_{\mathtt{EAL}^{\mathtt{A}}_{\mathtt{P}}}$-MCSs;
- $sR^c_a t \iff$ for any $\phi \in t$ then $\langle a\rangle\phi \in s \iff$ for any $[a]\phi \in s$ then $\phi \in t$;
- $s \sim^c t \iff$ for any $\phi \in t$ then $\hat{K}\phi \in s \iff$ for any $K\phi \in s$ then $\phi \in t$;
- $V^c(p) = \{s \mid p \in s\}$.

According to the canonicity of axioms T, 4, and 5, we know that $\sim^c$ is indeed an equivalence relation on S^c. To verify that $\mathcal{C}$ is indeed an EM model, we need to verify that $s \sim^c t$ implies $e(s) = e(t)$.

Proposition 3.7 *In the canonical model $\mathcal{C}$, $a \in e(s) \iff \langle a\rangle\top \in s$.*

Proof $\Rightarrow$: If $a \in e(s)$, according to the definition of $e(s)$, there is a $t \in S^c, s \xrightarrow{a} t$,because $\top \in t$, then $\langle a\rangle\top \in s$.

$\Leftarrow$: Let $D = \{\phi \mid [a]\phi \in s\}$. Since $\vdash [a](\phi \wedge \psi) \leftrightarrow [a]\phi \wedge [a]\psi$, s is closed under finite conjunctions. If D is not consistent, then there is $\phi \in D, \vdash \phi \to \bot$. By the rule NEC$(a)$, $\vdash [a](\phi \to \bot)$, thus by DIST$(a)$, $\vdash [a]\phi \to [a]\bot$, namely $\vdash [a]\phi \to \neg\langle a\rangle\top$. Since $[a]\phi \in s$, $\neg\langle a\rangle\top \in s$ which is contradictory to $\langle a\rangle\top \in s$. Therefore there is a maximal consistent set t such that $D \subseteq t$. According to the definition of R^c_a, we have $sR^c_a t$ thus $a \in e(s)$. □

Proposition 3.8 *In the canonical model $\mathcal{C}$, if $s \sim^c t$ then $e(s) = e(t)$.*

Proof For any $a \in \mathbf{A}$, if $a \in e(s)$, then according to Proposition 3.7, $\langle a\rangle\top \in s$. By axioms OBS$(a)$ and T, $K\langle a\rangle\top \in s$. Since $s \sim^c t$, $\langle a\rangle\top \in t$, by Proposition 3.7, $a \in e(t)$, namely $e(s) \subseteq e(t)$. It is symmetric to show $e(t) \subseteq e(s)$. □

In the rest of this section, we will show that the two semantics coincide on $\mathcal{C}$. To prove this, the key idea is to show that the equivalence classes of $\mathcal{C}$ capture all the possible dynamics of the uncertainty sets, e.g., if you move from a state s in an equivalence class U_s in $\mathcal{C}$ to a state t, then the updated uncertainty set $(U_s)|^a_t$ is exactly the equivalence class that t belongs to in $\mathcal{C}$. Formally, we have the following proposition (recall that $U_s = \{t \mid s \sim^c t \text{ in } \mathcal{C}\}$).

Proposition 3.9 *In the canonical model $\mathcal{C}$, if $s \xrightarrow{a} t$, then $(U_s)|^a \cap E(t) = U_t$. Namely $U_s|^a_t = U_t$ and thus $\mathcal{C}_s|^a_t = \mathcal{C}_t$.*

Proof $\subseteq$: Assuming $v \in (U_s)|^a \cap E(t)$, we need to prove that $v \in U_t$, namely $v \sim^c t$. Since $v \in (U_s)|^a$, there is a u, such that $u \sim^c s$ and $u \xrightarrow{a} v$. Let $\mathbf{B} = \{a \mid a \in \mathbf{A} \text{ and } \langle a\rangle\top \in v\}$, then $\psi_{\mathbf{B}} \in v$. For any $\phi \in v$, it is clear that $\phi \wedge \psi_{\mathbf{B}} \in v$. Since $u \xrightarrow{a} v$, $\langle a\rangle(\phi \wedge \psi_{\mathbf{B}}) \in u$. By $u \sim^c s$ we have $\hat{K}\langle a\rangle(\phi \wedge \psi_{\mathbf{B}}) \in s$. Now by axiom ONM$(a)$ and rule SUB, $[a](\psi_{\mathbf{B}} \to \hat{K}\phi) \in s$. Since $s \xrightarrow{a} t$, $\psi_{\mathbf{B}} \to \hat{K}\phi \in t$. Because $v \in E(t)$, then $\psi_{\mathbf{B}} \in t$ thus $\hat{K}\phi \in t$. By the definition of $\sim^c$, we have $v \sim^c t$, namely $v \in U_t$.

$\supseteq$: If $v \in U_t$, by Proposition 3.8, $e(v) = e(t)$ then $v \in E(t)$. In order to prove $v \in (U_s)|^a \cap E(t)$, we only need to show that $v \in (U_s)|^a$. In the following we will construct an MCS u such that $s \sim u$ and $u \xrightarrow{a} v$. Let $D = \{\psi \mid K\psi \in s\} \cup \{\langle a\rangle\phi \mid \phi \in v\}$. It is easy to see that $\{\psi \mid K\psi \in s\}$ is closed under finite conjunctions. If D is not consistent, we must have $\vdash \psi \wedge \langle a\rangle\phi_1 \wedge \cdots \wedge \langle a\rangle\phi_n \to \bot$ for some $K\psi \in s$ and $\phi_1 \ldots \phi_n \in v$, then $\vdash \psi \to ([a]\neg\phi_1 \vee \cdots \vee [a]\neg\phi_n)$. Because $\vdash [a]\neg\phi_1 \vee \cdots \vee [a]\neg\phi_n \to [a](\neg\phi_1 \vee \cdots \vee \phi_n)$, then $\vdash \psi \to [a](\neg\phi_1 \vee \cdots \vee \neg\phi_n)$. By NECK and DISTK, $\vdash K\psi \to K[a](\neg\phi_1 \vee \cdots \vee \neg\phi_n)$. By PR$(a)$ and SUB, $\vdash K[a](\neg\phi_1 \vee \cdots \vee \neg\phi_n) \to [a]K(\neg\phi_1 \vee \cdots \vee \neg\phi_n)$, then $\vdash K\psi \to [a]K(\neg\phi_1 \vee \cdots \vee \neg\phi_n)$. Since $K\psi \in s$, $[a]K(\neg\phi_1 \vee \cdots \vee \neg\phi_n) \in s$. Due to the fact that

$s \xrightarrow{a} t$, $K(\neg\phi_1 \vee \cdots \vee \neg\phi_n) \in t$. Since $v \sim^c t$, then $\neg\phi_1 \vee \cdots \vee \neg\phi_n \in v$. This is contradictory to $\phi_1 \ldots \phi_n \in v$ and that v is consistent. Therefore D is consistent, then there is a maximal consistent set u, such that $D \subseteq u$. Clearly $u \sim^c s$ and $u \xrightarrow{a} v$, thus $v \in (U_s)|^a$. In sum, $v \in (U_s)|^a \cap E(t)$. □

To prove the truth lemma w.r.t. $\vDash$ we make use of the following truth lemma w.r.t. $\Vdash$ as a standard exercise for normal modal logic (cf. e.g., [3]).

Lemma 3.10 *For any $\mathtt{EAL}^{\mathbf{A}}_{\mathbf{P}}$ formula ϕ, any s in $\mathcal{C}$: $\mathcal{C}, s \Vdash \phi \iff \phi \in s$.*

All we need now is to show that two semantics coincide on $\mathcal{C}$.

Lemma 3.11 *For any $\mathtt{EAL}^{\mathbf{A}}_{\mathbf{P}}$ formula ϕ, any s in $\mathcal{C}$: $\mathcal{C}, s \Vdash \phi \iff \mathcal{C}_s, s \vDash \phi$*

Proof Do induction on the structure of ϕ. The cases for $\phi = p$, $\phi = \neg\psi$, and $\phi = \phi_1 \wedge \phi_2$ are immediate.

$\phi = [a]\psi$, if $\mathcal{C}, s \Vdash [a]\psi$, but $\mathcal{C}_s, s \nvDash [a]\psi$, then there is a $t \in S^c$, such that $s \xrightarrow{a} t$ and $\mathcal{C}_s|^a_t, t \nvDash \psi$, by Proposition 3.9, $\mathcal{C}_s|^a_t = \mathcal{C}_t$, then $\mathcal{C}_t, t \nvDash \phi$, by IH, $\mathcal{C}, t \nVdash \phi$. Since $s \xrightarrow{a} t$, $\mathcal{C}, s \Vdash [a]\psi$, contradiction. On the other hand, if $\mathcal{C}_s, s \vDash [a]\psi$, but $\mathcal{C}, s \nVdash [a]\psi$, then there is a $t \in S^c$, such that $s \xrightarrow{a} t$, and $\mathcal{C}, t \nVdash \psi$. According to IH, $\mathcal{C}_t, t \nvDash \psi$. Now from Proposition 3.9, we have $\mathcal{C}_s|^a_t = \mathcal{C}_t$, then $\mathcal{C}_s|^a_t, t \nvDash \psi$. However, $s \xrightarrow{a} t$ and $\mathcal{C}_s, s \vDash [a]\psi$, contradiction.

$\phi = K\psi$, if $\mathcal{C}, s \Vdash K\psi$, but $\mathcal{C}_s, s \nvDash K\psi$, then there is a $u \in U_s$ such that $\mathcal{C}_s, u \nvDash \psi$. Since $u \in U_s$, then $U_s = U_u$, thus $\mathcal{C}_s = \mathcal{C}_u$, therefore $\mathcal{C}_u, u \nvDash \psi$. By IH $\mathcal{C}, u \nVdash \psi$ which is contradictory to the facts $u \sim^c s$ and $\mathcal{C}, s \Vdash K\psi$. On the other hand, if $\mathcal{C}_s, s \vDash K\phi$, but $\mathcal{C}, s \nVdash K\psi$, then there is a u, such that $s \sim^c u$ and $\mathcal{C}, u \nVdash \psi$. By IH, $\mathcal{C}_u, u \nvDash \psi$. Since $s \sim^c u$, then $U_s = U_u$ thus $\mathcal{C}_s = \mathcal{C}_u$. Therefore $\mathcal{C}_s, u \nvDash \psi$ which is contradictory to $u \in U_s$ and $\mathcal{C}_s, s \vDash K\psi$. □

Based on Lemmata 3.10 and 3.11, every $\mathbf{S}_{\mathtt{EAL}^{\mathbf{A}}_{\mathbf{P}}}$-consistent set of formulas has a model $\mathcal{C}_s, s$, thus the completeness is immediate.

Theorem 3.12 *$\mathbf{S}_{\mathtt{EAL}^{\mathbf{A}}_{\mathbf{P}}}$ is sound and complete on UM models.*

4 Modal theoretical properties of $\mathtt{EAL}^{\mathbf{A}}_{\mathbf{P}}$

In this section, we prove three results which can help us to understand $\mathtt{EAL}^{\mathbf{A}}_{\mathbf{P}}$ on UM models better. We first show that $\mathtt{EAL}^{\mathbf{A}}_{\mathbf{P}}$ is invariant under a special notion of bisimulation between UM models, which inspires a normal form theorem as our second result, and finally we prove the finite model property of $\mathtt{EAL}^{\mathbf{A}}_{\mathbf{P}}$ based on these insights.

4.1 Structural invariance for $\mathtt{EAL}^{\mathbf{A}}_{\mathbf{P}}$

Recall that given a UM model $\mathcal{M} = \langle S, \{R_a \mid a \in \mathbf{A}\}, U, V\rangle$, $\mathcal{M}^{\mathtt{ML}}$ is the multimodal model without U, i.e. $\mathcal{M}^{\mathtt{ML}} = \langle S, \{R_a \mid a \in \mathbf{A}\}, V\rangle$. Let $\mathtt{ML}^{\mathbf{A}}_{\mathbf{P}}$ be the K-free fragment of $\mathtt{EAL}^{\mathbf{A}}_{\mathbf{P}}$.

Next, we define a structural relation between UM models based on the notion of bisimilarity ($\leftrightarroweq$) on multimodal models w.r.t. **P** and **A** (cf. e.g., [3]).

Definition 4.1 For any UM models $\mathcal{M}$ and $\mathcal{N}$, we say that $\mathcal{M}$ is *U-bisimilar* to $\mathcal{N}$ (notation: $\mathcal{M} \rightleftarrows \mathcal{N}$) iff:

- for any $u \in U_{\mathcal{M}}$, there is a $u' \in U_{\mathcal{N}}$, such that $\mathcal{M}^{\mathtt{ML}}, u \leftrightarrows \mathcal{N}^{\mathtt{ML}}, u'$,
- for any $u' \in U_{\mathcal{N}}$, there is a $u \in U_{\mathcal{M}}$, such that $\mathcal{M}^{\mathtt{ML}}, u \leftrightarrows \mathcal{N}^{\mathtt{ML}}, u'$.

We say two *pointed* UM models are U-bisimilar ($\mathcal{M}, u \rightleftarrows \mathcal{N}, u'$) iff $\mathcal{M}^{\mathtt{ML}}, u \leftrightarrows \mathcal{N}^{\mathtt{ML}}, u'$ and $\mathcal{M} \rightleftarrows \mathcal{N}$.

Now we prove that the moves in UM models preserve U-bisimilarity.

Proposition 4.2 *If $\mathcal{M}, s \rightleftarrows \mathcal{N}, u$, $s \xrightarrow{a} t$ in $\mathcal{M}$, $u \xrightarrow{a} v$ in $\mathcal{N}$, and $\mathcal{M}^{ML}, t \leftrightarrows \mathcal{N}^{ML}, v$, then $\mathcal{M}|_t^a, t \rightleftarrows \mathcal{N}|_v^a, v$.*

Proof Since $(\mathcal{M}|_t^a)^{\mathtt{ML}} = \mathcal{M}^{\mathtt{ML}}, (\mathcal{N}|_v^a)^{\mathtt{ML}} = \mathcal{N}^{\mathtt{ML}}$ and $\mathcal{M}^{\mathtt{ML}}, t \leftrightarrows \mathcal{N}^{\mathtt{ML}}, v$, then by the definition of $\rightleftarrows$, we only need to prove that $\mathcal{M}|_t^a \rightleftarrows \mathcal{N}|_v^a$, namely for each x' in $U_{\mathcal{M}}|_t^a$ there is a y' in $U_{\mathcal{N}}|_v^a$ such that $(\mathcal{M}|_t^a)^{\mathtt{ML}}, x' \leftrightarrows (\mathcal{N}|_v^a)^{\mathtt{ML}}, y'$ (the reverse condition can be proved symmetrically).

For each $x' \in U_{\mathcal{M}}|_t^a = U_{\mathcal{M}}|^a \cap E(t)$, there is an $x \in U_{\mathcal{M}}$ and $x \xrightarrow{a} x'$ in $\mathcal{M}$. Since $\mathcal{M} \rightleftarrows \mathcal{N}$, then there is a $y \in U_{\mathcal{N}}$, such that $\mathcal{M}^{\mathtt{ML}}, x \leftrightarrows \mathcal{N}^{\mathtt{ML}}, y$. Since $x \xrightarrow{a} x'$ in $\mathcal{M}$, then according to the definition of bisimilarity, there is a $y' \in S_{\mathcal{N}}$, such that $y \xrightarrow{a} y'$ in $\mathcal{N}$ and $\mathcal{M}^{\mathtt{ML}}, x' \leftrightarrows \mathcal{N}^{\mathtt{ML}}, y'$ (thus $(\mathcal{M}|_t^a)^{\mathtt{ML}}, x' \leftrightarrows (\mathcal{N}|_v^a)^{\mathtt{ML}}, y'$.). Clearly $y' \in U_{\mathcal{N}}|^a$. We only need to show that $y' \in E(v)$ in order to prove that $y' \in U_{\mathcal{N}}|_v^a$. Since $\mathcal{M}^{\mathtt{ML}}, x' \leftrightarrows \mathcal{N}^{\mathtt{ML}}, y'$, $e(y') = e(x')$. Now due to the fact that $x' \in U_{\mathcal{M}}|_t^a$, we have $e(x') = e(t)$. Since $\mathcal{M}^{\mathtt{ML}}, t \leftrightarrows \mathcal{N}^{\mathtt{ML}}, v$, then it is easy to see that $e(t) = e(v)$, thus $e(y') = e(x') = e(t) = e(v)$, therefore $y' \in E(v)$, namely $y' \in U_{\mathcal{N}}|_v^a$. □

We say a UM model $\mathcal{M}$ is *image-finite* if $U_{\mathcal{M}}$ is finite, and for any $s \in S_{\mathcal{M}}$ and any $a \in \mathbf{A}$: $\{t \mid s \xrightarrow{a} t\}$ is finite. Let $\equiv_{\mathtt{EAL}_\mathbf{P}^\mathbf{A}}$ be the logical equivalence relation between pointed models. As an easy exercise, we can show that U-bisimilarity indeed preserves the truth values of the $\mathtt{EAL}_\mathbf{P}^\mathbf{A}$-formulas based on Proposition 4.2 (we omit the proof due to limited space).

Proposition 4.3 *For any pointed models $\mathcal{M}, u$, $\mathcal{N}, u'$: $\mathcal{M}, u \rightleftarrows \mathcal{N}, u'$ implies $\mathcal{M}, u \equiv_{\mathtt{EAL}_\mathbf{P}^\mathbf{A}} \mathcal{N}, u'$. The converse holds when restricted to image-finite models.*

4.2 Normal form

Proposition 4.3 says that the distinguishing power of $\mathtt{EAL}_\mathbf{P}^\mathbf{A}$ is bounded by the U-bisimilarity. A closer look reveals something more surprising: qua expressive power, the full language of $\mathtt{EAL}_\mathbf{P}^\mathbf{A}$ is equivalent to its fragment where knowledge operator only appears outside the scopes of the action modalities. Formally, formulas ϕ in this fragment ($\mathtt{EALK}_\mathbf{P}^\mathbf{A}$) can be generated by:

$$\begin{array}{rcl}\phi & ::= & \top \mid p \mid \psi \mid \neg\phi \mid \phi \wedge \phi \mid K\phi \\ \psi & ::= & \top \mid p \mid \neg\psi \mid \psi \wedge \psi \mid [a]\psi\end{array}$$

where $a \in \mathbf{A}$ and $p \in \mathbf{P}$.

In this subsection we will show that every $\mathtt{EAL}_\mathbf{P}^\mathbf{A}$ formula is equivalent to an (exponentially longer) $\mathtt{EALK}_\mathbf{P}^\mathbf{A}$ formula. Note that although Proposition 4.3 already suggests that $\mathtt{EAL}_\mathbf{P}^\mathbf{A}$ and $\mathtt{EALK}_\mathbf{P}^\mathbf{A}$ have the same distinguishing power, their expressive powers may still differ,[7] thus the result does not follow from Proposition 4.3.

Definition 4.4 We define the K-degree of $\mathtt{EAL}^{\mathbf{A}}_{\mathbf{P}}$ formulas ($kd(\phi)$) as follows:

$$\begin{array}{rlrl} kd(\top) &= 0 & kd(p) &= 0 \\ kd(\neg\phi) &= kd(\phi) & kd(\phi\wedge\psi) &= max\{kd(\phi), kd(\psi)\} \\ kd([a]\phi) &= 0 & kd(K\phi) &= 1 + kd(\phi) \end{array}$$

where $p \in \mathbf{P}$ and $a \in \mathbf{A}$.

Note that we treat the outmost $[\cdot]\phi$ (not in the scope of any other $[\cdot]$) as atomic formulas by setting $kd([\cdot]\phi) = 0$, e.g., $kd(K[a]K[b]p) = 1$.

Definition 4.5 An $\mathtt{EAL}^{\mathbf{A}}_{\mathbf{P}}$ formula ϕ is in *K-conjunctive normal form* (K-CNF) iff:

- $\phi = \alpha_1 \wedge \cdots \wedge \alpha_n$ such that $\forall 1 \le i \le n : \alpha_i = \beta_{i_1} \vee \cdots \vee \beta_{i_m}$ for some $m \ge 1$,
- each β_{i_j} is in the shape of $p, \neg p, [\cdot]\psi, \neg[\cdot]\psi$, $K\chi$ or $\hat{K}\chi$ where $kd(\chi) = 0$.

Note that $\mathbf{S}_{\mathtt{EAL}^{\mathbf{A}}_{\mathbf{P}}}$ includes all the axioms and rules of **S5**. Thus by using the standard result for **S5** logic, we can turn each $\mathtt{EAL}^{\mathbf{A}}_{\mathbf{P}}$ formula into K-CNF.

Proposition 4.6 *For any $\mathtt{EAL}^{\mathbf{A}}_{\mathbf{P}}$-formula ϕ, there is an $\mathtt{EAL}^{\mathbf{A}}_{\mathbf{P}}$ formula ϕ', such that $\vDash \phi \leftrightarrow \phi'$ and ϕ' is in K-CNF. In particular, for each $\mathtt{EALK}^{\mathbf{A}}_{\mathbf{P}}$-formula there is an equivalent $\mathtt{EALK}^{\mathbf{A}}_{\mathbf{P}}$-formula in K-CNF.*

Proof We treat the outmost $[\cdot]\psi$ formulas as atomic formulas when massaging the original formula according to the standard normal form result for **S5** (cf. e.g., [12]), thus keeping the formulas in $\mathtt{EALK}^{\mathbf{A}}_{\mathbf{P}}$. □

Note that in an $\mathtt{EAL}^{\mathbf{A}}_{\mathbf{P}}$ formula of K-CNF, there may still be some occurrences of K modality inside the scope of $[\cdot]$ modalities. In the sequel, we will try to *push* the K operator out. Here are two crucial results proved using the spirit behind the validity of axioms $\mathtt{PR}(\cdot)$ and $\mathtt{ONM}(\cdot)$.[8]

Proposition 4.7

$$(1) \vDash [a](K\phi \vee \chi) \leftrightarrow \bigwedge_{\boldsymbol{B}\subseteq\boldsymbol{A}} (\langle a\rangle(\psi_{\boldsymbol{B}} \wedge \neg\chi) \to K[a](\psi_{\boldsymbol{B}} \to \phi))$$

$$(2) \vDash [a](\hat{K}\phi \vee \chi) \leftrightarrow \bigwedge_{\boldsymbol{B}\subseteq\boldsymbol{A}} (\langle a\rangle(\psi_{\boldsymbol{B}} \wedge \neg\chi) \to \hat{K}\langle a\rangle(\psi_{\boldsymbol{B}} \wedge \phi))$$

Proof

(1) Left to right: If $\mathcal{M}, s \vDash [a](K\phi\vee\chi)$, we need to prove that for any $\mathbf{B} \subseteq \mathbf{A}$, $\mathcal{M}, s \vDash \langle a\rangle(\psi_{\mathbf{B}}\wedge\neg\chi) \to K[a](\psi_{\mathbf{B}} \to \phi)$. Now suppose $\mathcal{M}, s \vDash \langle a\rangle(\psi_{\mathbf{B}}\wedge\neg\chi)$, then

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[7] Here by *distinguishing power* we mean the power of a language to tell two models apart while expressive power measures the power of the language to define classes of models (properties of the models).

[8] The equivalences can be proved in $\mathbf{S}_{\mathtt{EAL}^{\mathbf{A}}_{\mathbf{P}}}$ too.

there is a t, such that $s \xrightarrow{a} t$, and $\mathcal{M}|_t^a, t \vDash \psi_{\mathbf{B}} \wedge \neg\chi$. Because $\mathcal{M}, s \vDash [a](K\phi \vee \chi)$, then $\mathcal{M}|_t^a, t \vDash K\phi \vee \chi$, then $\mathcal{M}|_t^a, t \vDash \psi_{\mathbf{B}} \wedge K\phi$.

We need to prove that $\mathcal{M}, s \vDash K[a](\psi_{\mathbf{B}} \rightarrow \phi)$, namely for any $u \in U_{\mathcal{M}}$ we need to show $\mathcal{M}, u \vDash [a](\psi_{\mathbf{B}} \rightarrow \phi)$. That is, for any v such that $u \xrightarrow{a} v$, we need to show $\mathcal{M}|_v^a, v \vDash \psi_{\mathbf{B}} \rightarrow \phi$. Suppose $\mathcal{M}|_v^a, v \vDash \psi_{\mathbf{B}}$, we then have $E(v) = E(t)$ since $\mathcal{M}|_t^a, t \vDash \psi_{\mathbf{B}}$. Therefore $U_{\mathcal{M}}|_t^a = U_{\mathcal{M}}|^a \cap E(t) = U_{\mathcal{M}}|^a \cap E(v) = U_{\mathcal{M}}|_v^a$, thus $\mathcal{M}|_t^a = \mathcal{M}|_v^a$ and $v \in U_{\mathcal{M}}|_t^a$. Since $\mathcal{M}|_t^a, t \vDash K\phi$, we have $\mathcal{M}|_t^a, v \vDash \phi$, thus $\mathcal{M}|_v^a, v \vDash \phi$.

(1) Right to left: Suppose for any $\mathbf{B} \subseteq \mathbf{A}$, $\mathcal{M}, s \vDash \langle a\rangle(\psi_{\mathbf{B}} \wedge \neg\chi) \rightarrow K[a](\psi_{\mathbf{B}} \rightarrow \phi)$, we need to show that $\mathcal{M}, s \vDash [a](K\phi \vee \chi)$. Suppose not, then there is a t, such that $s \xrightarrow{a} t$ and $\mathcal{M}|_t^a, t \vDash \neg K\phi \wedge \neg\chi \wedge \psi_{\mathbf{B}}$ for some $\mathbf{B} = e(t)$. Then $\mathcal{M}, s \vDash \langle a\rangle(\psi_{\mathbf{B}} \wedge \neg\chi)$. Since $\mathcal{M}, s \vDash \langle a\rangle(\psi_{\mathbf{B}} \wedge \neg\chi) \rightarrow K[a](\psi_{\mathbf{B}} \rightarrow \phi)$, then $\mathcal{M}, s \vDash K[a](\psi_{\mathbf{B}} \rightarrow \phi)$. Since $\mathcal{M}|_t^a, t \vDash \neg K\phi$, then there is $v \in U_{\mathcal{M}}|_t^a$, such that $\mathcal{M}|_t^a, v \vDash \neg\phi$ $(*)$. Since $v \in U_{\mathcal{M}}|_t^a$, then $v \in E(t)$ and there is a $u \in U_{\mathcal{M}}$ such that $u \xrightarrow{a} v$. Since $\mathcal{M}, s \vDash K[a](\psi_{\mathbf{B}} \rightarrow \phi)$, then $\mathcal{M}, u \vDash [a](\psi_{\mathbf{B}} \rightarrow \phi)$. Since $u \xrightarrow{a} v$, $\mathcal{M}|_v^a, v \vDash \psi_{\mathbf{B}} \rightarrow \phi$. Since $v \in E(t)$ and $\mathbf{B} = e(t)$, we have $\mathcal{M}|_v^a, v \vDash \psi_{\mathbf{B}}$ thus $\mathcal{M}|_v^a, v \vDash \phi$. Again by the fact that $v \in E(t)$ we have $U_{\mathcal{M}}|_t^a = U_{\mathcal{M}}|_v^a$, thus $\mathcal{M}|_t^a = \mathcal{M}|_v^a$. Now it is easy to see that $\mathcal{M}|_t^a, v \vDash \phi$ which is contradictory to $(*)$. Thus there is no such t that $s \xrightarrow{a} t$ and $\mathcal{M}|_t^a, t \vDash \neg K\phi \wedge \neg\chi$, therefore $\mathcal{M}, s \vDash [a](K\phi \vee \chi)$.

(2) Left to right: Assuming $\mathcal{M}, s \vDash [a](\hat{K}\phi \vee \chi)$, we need to prove that for any $\mathbf{B} \subseteq \mathbf{A}$: $\mathcal{M}, s \vDash \langle a\rangle(\psi_{\mathbf{B}} \wedge \neg\chi) \rightarrow \hat{K}\langle a\rangle(\psi_{\mathbf{B}} \wedge \phi)$.

Now suppose $\mathcal{M}, s \vDash \langle a\rangle(\psi_{\mathbf{B}} \wedge \neg\chi)$, then there is a t, such that $s \xrightarrow{a} t$ and $\mathcal{M}|_t^a, t \vDash \psi_{\mathbf{B}} \wedge \neg\chi$. Since $\mathcal{M}, s \vDash [a](\hat{K}\phi \vee \chi)$, then $\mathcal{M}|_t^a, t \vDash \hat{K}\phi \vee \chi$, then $\mathcal{M}|_t^a, t \vDash \psi_{\mathbf{B}} \wedge \hat{K}\phi$. Thus there is a $v \in U_{\mathcal{M}}|_t^a$, such that $\mathcal{M}|_t^a, v \vDash \phi$. Since $v \in U_{\mathcal{M}}|_t^a$, there is a $u \in U_{\mathcal{M}}$ such that $u \xrightarrow{a} v$ and $v \in E(t)$. $v \in E(t)$ implies that $\mathcal{M}|_t^a, v \vDash \psi_{\mathbf{B}}$. Then $\mathcal{M}|_t^a, v \vDash \psi_{\mathbf{B}} \wedge \phi$. Because $v \in E(t)$, then $U_{\mathcal{M}}|_t^a = U_{\mathcal{M}}|_v^a$, thus $\mathcal{M}|_t^a = \mathcal{M}|_v^a$. Therefore $\mathcal{M}|_v^a, v \vDash \psi_{\mathbf{B}} \wedge \phi$, and then we have $\mathcal{M}, u \vDash \langle a\rangle(\psi_{\mathbf{B}} \wedge \phi)$. Since $s, u \in U_{\mathcal{M}}$, $\mathcal{M}, s \vDash \hat{K}\langle a\rangle(\psi_{\mathbf{B}} \wedge \phi)$.

(2) Right to left: Suppose for any $\mathbf{B} \subseteq \mathbf{A}$, $\mathcal{M}, s \vDash \langle a\rangle(\psi_{\mathbf{B}} \wedge \neg\chi) \rightarrow \hat{K}\langle a\rangle(\psi_{\mathbf{B}} \wedge \phi)$ but $\mathcal{M}, s \nvDash [a](\hat{K}\phi \vee \chi)$, then there is a t, such that $s \xrightarrow{a} t$ and $\mathcal{M}|_t^a, t \vDash K\neg\phi \wedge \neg\chi \wedge \psi_{\mathbf{B}}$ for some $\mathbf{B} = e(t) \subseteq \mathbf{A}$. Therefore $\mathcal{M}, s \vDash \langle a\rangle(\psi_{\mathbf{B}} \wedge \neg\chi)$, thus $\mathcal{M}, s \vDash \hat{K}\langle a\rangle(\psi_{\mathbf{B}} \wedge \phi)$ due to the assumption that $\mathcal{M}, s \vDash \langle a\rangle(\psi_{\mathbf{B}} \wedge \neg\chi) \rightarrow \hat{K}\langle a\rangle(\psi_{\mathbf{B}} \wedge \phi)$. Now there is a $u \in U_{\mathcal{M}}$ such that $\mathcal{M}, u \vDash \langle a\rangle(\psi_{\mathbf{B}} \wedge \phi)$. Then there is a v, $u \xrightarrow{a} v$ and $\mathcal{M}|_v^a, v \vDash \psi_{\mathbf{B}} \wedge \phi$. Since $\mathcal{M}|_v^a, v \vDash \psi_{\mathbf{B}}$ and $e(t) = \mathbf{B}$, we have $v \in E(t)$, thus $U_{\mathcal{M}}|_t^a = U_{\mathcal{M}}|_v^a$. Therefore $\mathcal{M}|_t^a = \mathcal{M}|_v^a$, then $\mathcal{M}|_t^a, v \vDash \phi$ which contradicts to $\mathcal{M}|_t^a, t \vDash K\neg\phi$. Thus there is no such t that $s \xrightarrow{a} t$ and $\mathcal{M}|_t^a, t \vDash K\neg\phi \wedge \neg\chi$, therefore $\mathcal{M}, s \vDash [a](\hat{K}\phi \vee \chi)$. □

Now we are ready to prove the main theorem of this subsection.

Theorem 4.8 *For any $EAL_{\mathbf{P}}^{\mathbf{A}}$ formula ϕ, there is an $EALK_{\mathbf{P}}^{\mathbf{A}}$ formula ϕ', such that* $\vDash \phi \leftrightarrow \phi'$.

Proof Before we start the proof, note that by Proposition 3.6 and Theorem 3.12, the replacements of the equals preserve validity. We will use it repeatedly. We prove the theorem by induction on the structure of ϕ:

The cases for $\phi = p, \neg\phi', \phi_1 \wedge \phi_2$, and $K\phi'$ can be easily proved by IH and the replacement of equals.

$\phi = [a]\psi$, by IH, there is an $\mathtt{EALK}^{\mathbf{A}}_{\mathbf{P}}$-formula ψ', such that $\vDash \psi \leftrightarrow \psi'$. By Proposition 4.6, there is an $\mathtt{EALK}^{\mathbf{A}}_{\mathbf{P}}$-formula χ in K-CNF, such that $\vDash \chi \leftrightarrow \psi'$. Since χ is in K-CNF, then we can assume that $\chi = \alpha_1 \wedge \cdots \wedge \alpha_n$, then $[a]\chi$ is clearly equivalent to $[a]\alpha_1 \wedge \cdots \wedge [a]\alpha_n$. We want to show that for each $1 \leq i \leq n$, $[a]\alpha_i$ is equivalent to an $\mathtt{EALK}^{\mathbf{A}}_{\mathbf{P}}$-formula since then $[a]\alpha_1 \wedge \cdots \wedge [a]\alpha_n$ is also equivalent to an $\mathtt{EALK}^{\mathbf{A}}_{\mathbf{P}}$-formula.

Since χ is an $\mathtt{EALK}^{\mathbf{A}}_{\mathbf{P}}$-formula, each α_i is also an $\mathtt{EALK}^{\mathbf{A}}_{\mathbf{P}}$-formula. By the definition of K-CNF, each α_i is in the shape of $\beta_1 \vee \cdots \vee \beta_m$, then $[a]\alpha_i = [a](\beta_1 \vee \cdots \vee \beta_m)$. It is clear that for any $1 \leq j \leq m$, β_j is an $\mathtt{EALK}^{\mathbf{A}}_{\mathbf{P}}$-formula. By definition of K-CNF, each β_j is in the shape of $p, \neg p, [\cdot]\psi, \neg[\cdot]\psi$, $K\chi$ or $\hat{K}\chi$ where $kd(\chi) = 0$. Note that for $\mathtt{EALK}^{\mathbf{A}}_{\mathbf{P}}$- formulas $\phi : kd(\phi) = 0 \iff \phi$ is K-free. Now since β_j is in $\mathtt{EALK}^{\mathbf{A}}_{\mathbf{P}}$, then it is not hard to see that β_j contains K operator iff $\beta_j = K\chi$ or $\beta_j = \hat{K}\chi$ where χ is K-free. Then we can sort all the β_j into two categories depending on whether it is K-free, and rearrange the disjuncts in α_i as $\beta_{i_1} \vee \cdots \vee \beta_{i_h} \vee \cdots \vee \beta_{i_m}$, such that $kd(\beta_{i_k}) = 1$ for $1 \leq k \leq h$ and $kd(\beta_{i_k}) = 0$ for $h < k \leq m$. We will prove the following claim ($\star$):

For any $h \geq 0$ and any $m > h$ there is an $\mathtt{EALK}^{\mathbf{A}}_{\mathbf{P}}$-formula γ, such that $\vDash \gamma \leftrightarrow [a](\beta_{i_1} \vee \cdots \vee \beta_{i_h} \vee \cdots \vee \beta_{i_m})$.

We prove it by induction on h. The case of $h = 0$ is trivial since all the β_{i_k} ($1 \leq k \leq m$) are K-free.

Now suppose the claim holds when $h = n$ (for all $m > h$) we need to prove the case of $h = n + 1$. Let $\chi = \beta_{i_1} \vee \cdots \vee \beta_{i_n} \vee \beta_{i_{n+2}} \vee \cdots \vee \beta_{i_m}$, then $[a](\beta_{i_1} \vee \cdots \vee \beta_{i_m})$ is equivalent to $[a](\chi \vee \beta_{i_{n+1}})$. Since $kd(\beta_{i_{n+1}}) = 1$, thus $\beta_{i_{n+1}}$ contains K then $\beta_{i_{n+1}} = K\chi'$ or $\beta_{i_{n+1}} = \hat{K}\chi'$, where χ' is K-free.

(i) If $\beta_{i_{n+1}} = K\chi'$, then $[a](\chi \vee \beta_{i_{n+1}}) = [a](\chi \vee K\chi')$. By Proposition 4.7 (1), $[a](\chi \vee K\chi')$ is equivalent to

$$\bigwedge_{\mathbf{B} \subseteq \mathbf{A}} (\neg[a](\neg\psi_{\mathbf{B}} \vee \chi) \to K[a](\neg\psi_{\mathbf{B}} \vee \chi'))$$

Note that given an $\mathbf{B} \subseteq \mathbf{A}$, $\psi_{\mathbf{B}}$ is a K-free $\mathtt{EALK}^{\mathbf{A}}_{\mathbf{P}}$-formula in the shape of $\bigwedge_{a \in \mathbf{B}} \langle a \rangle \top \wedge \bigwedge_{b \notin \mathbf{B}} \neg\langle b \rangle \top$. Therefore $\neg\psi_{\mathbf{B}}$ is still an $\mathtt{EALK}^{\mathbf{A}}_{\mathbf{P}}$-formula which is equivalent to $\bigvee_{a \in \mathbf{B}} [a]\bot \vee \bigvee_{b \notin \mathbf{B}} \neg[b]\bot$. Therefore $\neg\psi_{\mathbf{B}} \vee \chi$ can be massaged into a right disjunctive form $\beta_{i_1} \vee \cdots \vee \beta_{i_n} \vee \cdots \vee \beta_{i_{m+|\mathbf{A}|}}$. Now by IH, there is an $\mathtt{EALK}^{\mathbf{A}}_{\mathbf{P}}$-formula $\gamma_{\mathbf{B}}$, such that $\gamma_{\mathbf{B}}$ is equivalent to $[a](\neg\psi_{\mathbf{B}} \vee \chi)$. Since χ' is K-free, then $K[a](\neg\psi_{\mathbf{B}} \vee \chi')$ is already an $\mathtt{EALK}^{\mathbf{A}}_{\mathbf{P}}$-formula. Now let $\theta_{\mathbf{B}} = \neg\gamma_{\mathbf{B}} \to K[a](\neg\psi_{\mathbf{B}} \vee \chi')$ we can see that $\theta_{\mathbf{B}}$ is an $\mathtt{EALK}^{\mathbf{A}}_{\mathbf{P}}$-formula equivalent to $\neg[a](\neg\psi_{\mathbf{B}} \vee \chi) \to K[a](\neg\psi_{\mathbf{B}} \vee \chi')$.

(ii) The case for $\beta_{i_{n+1}} = \hat{K}\chi'$ can be proved similarly by using Proposition 4.7 (2). Hereby we complete the proof for claim ($\star$).

In sum, for each i: $[a]\alpha_i$ is equivalent to an $\mathtt{EALK}^{\mathbf{A}}_{\mathbf{P}}$-formula thus $[a]\psi$ is equivalent to an $\mathtt{EALK}^{\mathbf{A}}_{\mathbf{P}}$-formula therefore completing the proof of the theorem. □

The above proof also suggests a naive algorithm to translate an $\mathtt{EAL}^{\mathbf{A}}_{\mathbf{P}}$-formula into an $\mathtt{EALK}^{\mathbf{A}}_{\mathbf{P}}$-formula, which works in the inside-out fashion:

(i) Find the minimal sub-formulas which are not in $\mathtt{EALK}^{\mathbf{A}}_{\mathbf{P}}$, massage them into K-CNF, and then use the method described in the above proof to translate them into equivalent $\mathtt{EALK}^{\mathbf{A}}_{\mathbf{P}}$-formulas by pushing K out.

(ii) Replacing those sub-formulas in the original formula by their $\mathtt{EALK}^{\mathbf{A}}_{\mathbf{P}}$ correspondents.

(iii) Repeat step (i) until all the subformulas are in the right shapes. Every step pushes the K operator one level out towards the outmost positions thus the procedure terminates eventually.

For example let $\mathbf{A} = \{a\}$, $[a]Kp$ can be translated to a K-CNF $\mathtt{EALK}^{\mathbf{A}}_{\mathbf{P}}$formula:

$$([a][a]\bot \vee K[a](\langle a\rangle\top \to p)) \wedge ([a]\langle a\rangle\top \vee K[a]([a]\bot \to p))$$

Let $\chi_1 = [a][a]\bot$, $\chi_2 = [a]\langle a\rangle\top$ and $\phi_1 = [a](\langle a\rangle\top \to p)$ and $\phi_2 = [a]([a]\bot \to p)$. Thus $[a][a]Kp$ is equivalent to: $[a](\chi_1 \vee K\phi_1) \wedge [a](\chi_2 \vee K\phi_2)$ and then to

$$\bigwedge_{i=1,2}(((\langle a\rangle(\langle a\rangle\top\wedge\neg\chi_i) \to K[a](\langle a\rangle\top \to \phi_i))\wedge(\langle a\rangle(\neg\langle a\rangle\top\wedge\neg\chi_i) \to K[a](\neg\langle a\rangle\top \to \phi_i)))$$

Clearly, the translated formula is at least exponentially longer. We leave the discussion on the succinctness of $\mathtt{EAL}^{\mathbf{A}}_{\mathbf{P}}$ compared to $\mathtt{EALK}^{\mathbf{A}}_{\mathbf{P}}$ for future work.

Remark 4.9 Is there a simpler translation? In many DEL-style logics, we can often define a simple recursive translation from the full language to its fragment by swapping the connectives and modalities, e.g., in public announcement logic, $[\psi]\neg\phi \iff \psi \to \neg[\psi]\phi$ (cf. e.g., [20]). However, such idea may not work here: it seems there is no general equivalence-preserving rule to swap $\langle a\rangle$ and $\neg$.

4.3 Finite model property

Theorem 4.8 also suggests that $\mathtt{EAL}^{\mathbf{A}}_{\mathbf{P}}$ has the *finite model property*: First of all, it is not hard to see that for any $\mathtt{EALK}^{\mathbf{A}}_{\mathbf{P}}$ formula ϕ, ϕ has a UM model iff ϕ has an EM model (w.r.t. $\Vdash$); Secondly, $\mathtt{EALK}^{\mathbf{A}}_{\mathbf{P}}$ on EM model has the finite model property (an easy exercise for normal modal logic); Thirdly any pointed EM model of an $\mathtt{EALK}^{\mathbf{A}}_{\mathbf{P}}$-formula can be viewed as an $\mathtt{EALK}^{\mathbf{A}}_{\mathbf{P}}$-equivalent UM model by ignoring the equivalence classes that do not contain the designated point.

In the rest of this section, we directly prove the finite model property of $\mathtt{EAL}^{\mathbf{A}}_{\mathbf{P}}$ on UM models by using finite approximations of U-bisimilarity.

Definition 4.10 The modal degree of $\mathtt{EAL}^{\mathbf{A}}_{\mathbf{P}}$-formulas ($md(\phi)$) is defined as follows:

$$\begin{array}{ll} md(\top) = 0 & md(p) = 0 \\ md(\neg\phi) = md(\phi) & md(\phi \wedge \psi) = max\{md(\phi), md(\psi)\} \\ md(\langle a\rangle\phi) = 1 + md(\phi) & md(K\phi) = md(\phi) \end{array}$$

Note that here K does not count for modal degrees. Let $\mathtt{EAL}^{\mathbf{A}}_{\mathbf{P}n} = \{\phi \mid \phi \in \mathtt{EAL}^{\mathbf{A}}_{\mathbf{P}}$, and $md(\phi) \leq n\}$

Now we define the finite approximation of $\rightleftarrows$ based on n-bisimilarity w.r.t. **P** and **A** (cf. [3]).

Definition 4.11 EM model $\mathcal{M}$ and $\mathcal{N}$ are *n-U-bisimilar* ($\mathcal{M} \rightleftarrows_n \mathcal{N}$) iff for any $u \in U_{\mathcal{M}}$, there is a $u' \in U_{\mathcal{N}}$ such that $\mathcal{M}^{\mathtt{ML}}, u \leftrightarroweq_n \mathcal{N}^{\mathtt{ML}}, u'$ and for any $u' \in U_{\mathcal{N}}$, there is a $u \in U_{\mathcal{M}}$ such that $\mathcal{M}^{\mathtt{ML}}, u \leftrightarroweq_n \mathcal{N}^{\mathtt{ML}}, u'$. For pointed models, $\mathcal{M}, s \rightleftarrows_n \mathcal{N}, u$ iff $\mathcal{M}^{\mathtt{ML}}, s \leftrightarroweq_n \mathcal{N}^{\mathtt{ML}}, u$ and $\mathcal{M} \rightleftarrows_n \mathcal{N}$.

Proposition 4.12 *For $n > 0$: if $\mathcal{M}, s \rightleftarrows_{n+1} \mathcal{N}, u$, $s \xrightarrow{a} t$, $u \xrightarrow{a} v$, and $\mathcal{M}^{ML}, t \leftrightarroweq_n \mathcal{N}^{ML}, v$, then $\mathcal{M}|^a_t, t \rightleftarrows_n \mathcal{N}|^a_v, v$.*

Proof Since $(\mathcal{M}|^a_t)^{\mathtt{ML}} = \mathcal{M}^{\mathtt{ML}}$ and $(\mathcal{N}|^a_v)^{\mathtt{ML}} = \mathcal{N}^{\mathtt{ML}}$, $(\mathcal{M}|^a_t)^{\mathtt{ML}}, t \leftrightarroweq_n (\mathcal{N}|^a_v)^{\mathtt{ML}}, v$, thus we only need to prove $\mathcal{M}|^a_t \rightleftarrows_n \mathcal{N}|^a_v$.

For any $t' \in U_{\mathcal{M}}|^a_t = U^a_{\mathcal{M}} \cap E(t)$, there is an $s' \in U_{\mathcal{M}}$, such that $s' \xrightarrow{a} t'$. Since $\mathcal{M}, s \rightleftarrows_{n+1} \mathcal{N}, u$, then there is a $u' \in U_{\mathcal{N}}$, such that $\mathcal{M}^{\mathtt{ML}}, s' \leftrightarroweq_{n+1} \mathcal{N}^{\mathtt{ML}}, u'$. Therefore there is a v', such that $u' \xrightarrow{a} v'$ and $\mathcal{M}^{\mathtt{ML}}, t' \leftrightarroweq_n \mathcal{N}^{\mathtt{ML}}, v'$. Therefore $v' \in U_{\mathcal{N}}|^a$ and $e(v') = e(t')$ (due to the fact that $n > 0$ and $\mathcal{M}^{\mathtt{ML}}, t' \leftrightarroweq_n \mathcal{N}^{\mathtt{ML}}, v'$). Since $e(t') = e(t)$ and $e(t) = e(v)$, we have $e(v') = e(v)$, thus $v' \in E(v)$, therefore $v' \in U_{\mathcal{N}}|^a_v$. Now we have proved that for any $t' \in U_{\mathcal{M}}|^a_t$, there is a $v' \in U_{\mathcal{N}}|^a_v$, such that $\mathcal{M}^{\mathtt{ML}}, t' \leftrightarroweq_n \mathcal{N}^{\mathtt{ML}}, v'$, namely $(\mathcal{M}|^a_t)^{\mathtt{ML}}, t \leftrightarroweq_n (\mathcal{N}|^a_v)^{\mathtt{ML}}, v$. The other direction is totally symmetric. □

Proposition 4.13 $\mathcal{M}, s \rightleftarrows_{n+1} \mathcal{N}, u \implies \mathcal{M}, s \equiv_{\mathtt{EAL}^{\mathbf{A}}_{\mathbf{P}n}} \mathcal{N}, u$.

Proof The proof is based on induction on n: For $n = 0$, we can easily check that all the $\mathtt{EAL}^{\mathbf{A}}_{\mathbf{P}0}$-formulas are preserved under $\rightleftarrows_1$. Now suppose $\mathcal{M}, s \rightleftarrows_{k+1} \mathcal{N}, u$ implies $\mathcal{M}, s \equiv_k \mathcal{N}, u$. We need to show $\mathcal{M}, s \rightleftarrows_{k+1+1} \mathcal{N}, u$ implies for all $\mathtt{EAL}^{\mathbf{A}}_{\mathbf{P}k+1}$ formula ϕ: $\mathcal{M}, s \vDash \phi \iff \mathcal{N}, u \vDash \phi$. Now suppose that $\mathcal{M}, s \rightleftarrows_{k+1+1} \mathcal{N}, u$, we proceed by induction on the structure ϕ:

For Boolean cases, it is obvious.

$\phi = [a]\psi$: If $\mathcal{N}, u \vDash [a]\psi$ but $\mathcal{M}, s \nvDash [a]\psi$, then there is a t, such that $s \xrightarrow{a} t$, and $\mathcal{M}|^a_t, t \nvDash \psi$. Since $\mathcal{M}, s \leftrightarroweq_{k+2} \mathcal{N}, u$, then there is a v, $u \xrightarrow{a} v$, and $\mathcal{M}^{\mathtt{ML}}, t \leftrightarroweq_{k+1} \mathcal{N}^{\mathtt{ML}}, v$. By Proposition 4.12, $\mathcal{M}|^a_t, t \rightleftarrows_{k+1} \mathcal{N}|^a_v, v$. Since $md(\psi) \leq k$, and by IH for k, $\mathcal{N}|^a_v, v \nvDash \psi$, this is contradictory to the fact that $\mathcal{N}, u \vDash [a]\psi$. The other direction is symmetric.

$\phi = K\psi$, if $\mathcal{N}, u \vDash K\psi$ but $\mathcal{M}, s \nvDash K\psi$ then there is a $s' \in U_{\mathcal{M}}$, $\mathcal{M}, s' \nvDash \psi$. Since $\mathcal{M}, s \rightleftarrows_{k+2} \mathcal{N}, u$ then there is a $u' \in U_{\mathcal{N}}$, $\mathcal{M}^{\mathtt{ML}}, s' \leftrightarroweq_{k+2} \mathcal{N}^{\mathtt{ML}}, u'$ then $\mathcal{M}, s' \rightleftarrows_{k+2} \mathcal{N}, u'$. By IH for simpler ϕ, $\mathcal{N}, u' \nvDash \psi$. Then $\mathcal{N}, u \nvDash K\psi$, contradiction. The other direction is symmetric. □

The reader may wonder about the mismatch between n and $n + 1$ in the above proposition. Actually it is not surprising since in the semantics we actually look one step forward to observe the available actions. The translation of the previous subsection also showed that the $\mathtt{EALK}^{\mathbf{A}}_{\mathbf{P}}$ equivalent translation may have larger modality depth than the original formula (check the example of $[a]Kp$ in the previous subsection). [9]

Theorem 4.14 *For each $\mathtt{EAL}^{\mathbf{A}}_{\mathbf{P}}$-formula ϕ, if it has a UM model then it has a finite tree-like model with the depth of at most $n+1$ where $n = md(\phi)$.*

Proof (Sketch:) Without loss of generality, we assume **P** and **A** are finite (since a formula is about at most finitely many symbols). Let $n = md(\phi)$. Suppose ϕ has a UM model $\mathcal{M}$, we first "contract" the set $U_{\mathcal{M}}$ according to $\leftrightarroweq_{n+1}$ (equivalently, according to $\equiv_{\mathtt{ML}_{n+1}}$, where $\mathtt{ML}_{n+1}$ is the K-free fragment of $\mathtt{EAL}^{\mathbf{A}}_{\mathbf{P}\,n+1}$). Note that since $\mathtt{ML}_{n+1}$ is essentially a finite language modulo logical equivalence, after the contraction there are only finitely many representatives which form a finite set U'. Then we finitely (up to $n+1$) unravel the pointed multimodal models based on these states and prune the branches to make a finite model. Finally we make a disjoint union of these unravellings with the uncertainty set U'. □

To squeeze the model even further we also develop a highly non-trivial filtration technique which we left for the full version of this paper. Based on such a finite model property and the fact that there is a finite complete axiomatization, we can conclude that $\mathtt{EAL}^{\mathbf{A}}_{\mathbf{P}}$ is decidable on UM models.

5 Comparisons

We claimed in the introduction that our $\mathtt{EAL}^{\mathbf{A}}_{\mathbf{P}}$ framework is a blend of ETL and DEL frameworks. In this section, we make it more precise by comparing $\mathtt{EAL}^{\mathbf{A}}_{\mathbf{P}}$ to ETL and DEL. The conclusions can be summarized as follows:

- Our UM models can be viewed as compact representations of ETL structures where the epistemic relations are computed based on the following: 1. the previous epistemic uncertainties, 2. the executed actions, and 3. the new observations, which is, in spirit, similar to the DEL-like epistemic updates.
- On the other hand, the DEL approach via product updates on epistemic models with protocols can also be viewed as a logic on particular ETL structures, however, satisfying a property which is violated in the navigation scenarios. Due to this and other difficulties, the standard DEL with event models is not suitable to handle the reasoning in the navigation scenarios.

To facilitate the comparisons, let us fix some notations first. Given a UM model $\mathcal{M}$ for $\mathtt{EAL}^{\mathbf{A}}_{\mathbf{P}}$, we say $\rho = s_0 a_1 s_1 a_2 s_2 \cdots a_n s_n$ is a *path* in $\mathcal{M}$ if $s_0 \in U_{\mathcal{M}}$, $n \geq 0$ and for any $0 \leq i \leq n-1$: $s_i \overset{a_{i+1}}{\to} s_{i+1}$ in $\mathcal{M}$. Given a path $\rho = s_0 a_1 s_1 a_2 s_2 \cdots a_n s_n$ let $len(\rho) = n$ be the *length* of ρ. Note that there are paths of length 0 consisting of a single state.

5.1 Comparison with ETL

Technically speaking, a single-agent ETL model is just a tree-like EM model. We can unravel a UM model into such an ETL model.

[9] Actually, a closer analysis would reveal that this can only happen when $md(\phi) = 1$. Proposition 4.13 can be strengthened to : for $n > 1$: $\mathcal{M}, s \rightleftarrows_n \mathcal{N}, u \iff \mathcal{M}, s \equiv_{\mathtt{EAL}^{\mathbf{A}}_{\mathbf{P}\,n}} \mathcal{N}, u$.

Definition 5.1 Given a UM model $\mathcal{M} = \langle S, \{R_a \mid a \in \mathbf{A}\}, U, V\rangle$, we define $\mathcal{M}^{\mathrm{ETL}}$ as $\langle S^\bullet, \{R^\bullet_a \mid a \in \mathbf{A}\}, \sim, V^\bullet\rangle$ where:

(i) $S^\bullet = \{\rho \mid \rho$ is a path in $\mathcal{M}$ starting with some $s \in U\}$

(ii) $\rho \xrightarrow{a} \rho'$ in $\mathcal{M}^{\mathrm{ETL}}$ iff $\rho' = \rho a t$ for some $t \in S$ and $a \in \mathbf{A}$.

(iii) For any two paths $\rho = s_0 a_1 \cdots a_n s_n$ and $\rho' = t_0 b_1 \cdots b_m t_m$ in $S^\bullet$: $\rho \sim \rho'$ in $\mathcal{M}^\bullet$ iff ($n = m$, for all $i \leq n$: $a_i = b_i$ and $e(s_i) = e(t_i)$).

(iv) $V^\bullet(s_0 a_1 \cdots a_n s_n) = V(s_n)$

It is easy to show that $\sim$ is indeed an equivalence relation. Moreover we can define $\sim$ more explicitly:

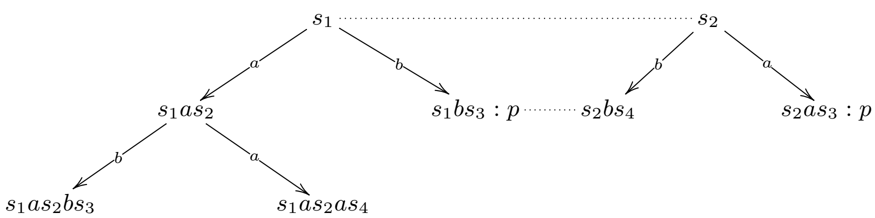
Proposition 5.2 $\sim \subseteq S^\bullet \times S^\bullet$ *is the minimal set of pairs satisfying the following conditions:*

(i) $s \sim t$ *for any* $s, t \in U$

(ii) $\rho a s \sim \rho' a' s'$ *if* $\rho \sim \rho'$, $a = a'$ *and* $e(s) = e(s')$.

Let us unravel the initial model in Example 2.1.

Example 5.3 Given the initial model as $\mathcal{M}$, $\mathcal{M}^{\mathrm{ETL}}$ can be depicted as follows (where dotted lines denote the $\sim$ relation while omitting the reflexive arrows):



The following result is crucial to prove that $\mathcal{M}^{\mathrm{ETL}}$ is indeed a good transformation preserving the truth values of $\mathtt{EAL}^{\mathbf{A}}_{\mathbf{P}}$ formulas.

Proposition 5.4 *Let* $\mathcal{M} = \langle S, \{R_a \mid a \in \boldsymbol{A}\}, U, V\rangle$ *and* $s \in U$. *If there exists an* $s' \in S$ *such that* $s \xrightarrow{a} s'$, *then* $\mathcal{M}^\bullet, sas' \leftrightarroweq (\mathcal{M}|^a_{s'})^\bullet, s'$ *(here the bisimilarity is w.r.t.* $\boldsymbol{P}$,$\boldsymbol{A}$ *and also* $\sim$*).*

Proof We define a binary relation Z on $S_{\mathcal{M}^\bullet} \times S_{(\mathcal{M}|^a_{s'})^\bullet}$ as follows $Z = \{(\rho, \rho') \mid \rho \in S_{\mathcal{M}^\bullet}, \rho' \in S_{(\mathcal{M}|^a_{s'})^\bullet}$ and there exists an $u \in U_{\mathcal{M}}$ such that $\rho = ua\rho'\}$.

Clearly $sas'Zs'$, thus Z is non-empty. Now we prove that Z is a bisimulation. The propositional invariance condition and the back-and-forth conditions for $\xrightarrow{a}$ are obvious. We only need to check the back-and-forth conditions for $\sim$. In the sequel, suppose $\rho = ua\rho'$ for some path ρ' in $S_{(\mathcal{M}|^a_{s'})^\bullet}$, then it is clear that ρ' starts with some state $t \in S_{\mathcal{M}}$ such that $e(t) = e(s')$.

Now suppose $\rho \sim \xi$ then according to the definition of $\sim$, ξ must be in the shape of $va\xi'$ where $v \in U_{\mathcal{M}}$ and ξ' must start with a state $t' \in S_{\mathcal{M}}$ such that $e(t') = e(t) = e(s')$. Therefore $t' \in U_{\mathcal{M}|^a_{s'}}$, then it is not hard to see that $\rho' \sim \xi'$ in $\mathcal{M}|^a_{s'}$. For the other direction, suppose $\rho' \sim \xi'$ in $\mathcal{M}|^a_{s'}$ then it is easy to see that there is a $v \in U_{\mathcal{M}}$ such that $ua\rho' \sim va\xi'$ by definition of $\sim$ in $\mathcal{M}$. $\square$

Recall that $\Vdash$ denotes the satisfaction relation of the auxiliary semantics of $\mathtt{EAL}^{\mathbf{A}}_{\mathbf{P}}$ language on EM models used in Section 3. The following preservation result can be proved based on Proposition 5.4 without much efforts.

Theorem 5.5 *For any $\mathtt{EAL}^{\mathbf{A}}_{\mathbf{P}}$ formula ϕ: $\mathcal{M}, s \vDash \phi \iff \mathcal{M}^{\bullet}, s \Vdash \phi$.*

Theorem 5.5 established the equivalence of our framework and ETL framework (restricted to the models with epistemic relations computed in a particular way). However, it is not reasonable to work with ETL models explicitly since the unravelling transforms a finite map into an infinite forest if there are loops.

5.2 Comparison with DEL

As we mentioned in the introduction, there are efforts trying to merge ETL and DEL frameworks. Most notably, [21] characterizes the DEL-generated ETL models (under protocols) by a few properties. In [6] the authors argue that *synchronicity* is not an inherent feature of DEL but is introduced by the specific translation used in [21]. However, it is usually agreed that the property of *(local) no miracles (LNM)* is inevitable for a DEL-generated ETL model.[10] Formally, LNM says that in the DEL-generated ETL model, for any states $s, s', t, t', w, w', v, v'$ and any action labels c, d:

$$((s \xrightarrow{c} s', t \xrightarrow{d} t', s' \sim t') \text{ and } (s \sim w \sim v, w \xrightarrow{c} w', v \xrightarrow{d} v')) \text{ implies } w' \sim v'.$$

In picture (whether s and t are indistinguishable is unknown):

$$\begin{tikzcd}
t \arrow[d, "d"] & s \arrow[d, "c"] \arrow[r, dotted, no head] & w \arrow[d, "c"] \arrow[r, dotted, no head] & v \arrow[d, "d"] \\
t' \arrow[r, dotted, no head] & s' & w' \arrow[r, dotted, no head] & v'
\end{tikzcd}$$

In words, LNM roughly says that whenever the results of executing two actions are indistinguishable, then executing these two actions on indistinguishable states must result in indistinguishable states again. Actually, this property is inherent in the definition of the updated epistemic relations according to the standard product update [1], where two updated states are indistinguishable after executing two actions on two old states respectively iff the two old states are indistinguishable and the two actions are indistinguishable too.

However, in our navigation scenarios, it is often the case that executing the same action (thus indistinguishable from itself) on previously indistinguishable states will result in distinguishable states, e.g., in Example 5.3, performing action a on indistinguishable states s_1 and s_2 will result in distinguishable states. To see that it is an example violating LNM, let $s = t = s_2, s' = t' = s_2 a s_3, c = d = a, w = s_1$, $v = s_2$ and $w' = s_1 a s_2, v' = s_2 a s_3$ and check the definition of LNM[11]. This is reflected syntactically in the difference between our $\mathtt{ONM}(\cdot)$

[10] Although we take the notion of LNM as in [21], our counterexample also works for the other no miracle notions in [6].

[11] Actually LNM also prevents the application of standard DEL in security verification as observed in [5].

axiom and the usual no miracle axiom schema $\hat{K}\langle a\rangle p \to [a]\hat{K}p$.

The feature of our epistemic update mechanism is reflected in Proposition 5.2: the three conditions in the inductive case actually say (1) the old uncertainties on states are respected; (2) we do not have uncertainties about actions with different names; (3) the observations at the new states may affect the new uncertainties. It is the feature (3) which makes us deviate from LNM.

Merely technically speaking, we may still try to mimic the $\mathtt{EAL}^{\mathbf{A}}_{\mathbf{P}}$ framework by the standard DEL via event models. The difficulties and the potential solutions are summarized below, interested readers may consult the algebraic DEL-approaches in [17,18,10].

- Failure of LNM: try to split one action into different actions w.r.t. different conditions, e.g., in Example 5.3 the two a moves must be treated differently in the event model.
- The standard DEL-model does not contain procedural information: use state-dependent protocols to encode the moves in the map (cf. e.g., [21]).
- No location changes: try to capture the changes of current location by factual changing actions (cf. e.g., [23,18]).
- DEL-updates are functional: to model non-deterministic actions, multi-pointed event models should be used.

6 Conclusion and future work

In this paper, we lay out a logical framework for dynamic epistemic reasoning in navigation. The "philosophy" behind our work is summarized as follows:

- Keep the logical language and its model as simple and natural as possible, while put the complexity on the semantics.
- Combine the spirits from ETL and DEL by having the temporal possibilities encoded in the model while computing the epistemic developments step by step according to the update semantics.
- When proving theoretical results, try to reduce the dynamics of the models into static relations in a larger model.

We think this is just the opening of an interesting story. A few future directions are mentioned as follows. For the current framework, we have not discussed the computational issues such as the complexity of satisfiability and model checking problems and the succinctness compared to $\mathtt{EALK}^{\mathbf{A}}_{\mathbf{P}}$. To generalize the current framework, we may consider general observations instead of observations of the currently available actions only, for which we may use propositional variables to encode available observations as in [22]. Similar techniques for axiomatization should work in the more general case. As in [18], the converse action operator may be introduced to express "I know where I was", although we conjecture that it can be eliminated qua expressive power. Finally it is natural to ask whether our framework can be extended to multi-agent.

As we mentioned in the introduction, we are aiming at real-life applications of navigation and planning, for which an epistemic Propositional Dynamic Logic (EPDL) language is more attractive due to its program language. We can then reduce the planning problem into model checking problem of EPDL-formulas expressing sentences like "there is a plan that can make sure he knows ϕ". This extension may require new techniques and we leave it for future work.

Acknowledgement This work is supported by SSFC grant 11CZX054. The first author would like to thank Dr. Mehrnoosh Sadrzadeh for pointing out the problem of robot navigation during her visit to Amsterdam in 2010. The authors are also grateful to the anonymous referees of AiML2012 for their insightful comments on the early version of this paper.

References

[1] Baltag, A. and L. Moss, *Logics for epistemic programs*, Synthese **139** (2004), pp. 165–224.

[2] Baltag, A., L. Moss and S. Solecki, *The logic of public announcements, common knowledge, and private suspicions*, in: *Proceedings of TARK'98* (1998), pp. 43–56.

[3] Blackburn, P., M. de Rijke and Y. Venema, "Modal Logic," Cambridge University Press, 2002.

[4] Bonet, B. and H. Geffner, *Planning with incomplete information as heuristic search in belief space*, in: *Artificial Intelligence Planning Systems*, 2000, pp. 52–61.

[5] Dechesne, F. and Y. Wang, *To know or not to know: Epistemic approaches to security protocol verification*, Synthese **177** (2010), pp. 51–76.

[6] Dégremont, C., B. Löwe and A. Witzel, *The synchronicity of dynamic epistemic logic*, in: *Proceedings of TARK'11*, 2011, pp. 145–152.

[7] Fagin, R., J. Halpern, Y. Moses and M. Vardi, "Reasoning about knowledge," MIT Press, Cambridge, MA, USA, 1995.

[8] Goldman, R. and M. Boddy, *Expressive planning and explicit knowledge*, in: *Proceedings of AIPS'96)*, 1996, pp. 110–117.

[9] Halpern, J., R. van der Meyden and M. Vardi, *Complete axiomatizations for reasoning about knowledge and time*, SIAM Journal on Computing **33** (2004), pp. 674–703.

[10] Horn, A., *Dynamic epistemic algebra with post-conditions to reason about robot navigation*, in: L. Beklemishev and R. de Queiroz, editors, *Proceedings of WoLLIC'11*, LNCS **6642** (2011), pp. 161–175.

[11] Hoshi, T., "Epistemic Dynamics and Protocol Information." Ph.D. thesis, Stanford (2009).

[12] Hughes, G. E. and M. J. Cresswell, "A New Introduction to Modal Logic," Routledge, 1996.

[13] Levesque, H. J., R. Reiter, Y. Lespérance, F. Lin and R. B. Scherl, *GOLOG: A logic programming language for dynamic domains*, Journal of Logic Programming **31** (1997), pp. 1–3.

[14] Meyer, J.-J. C., W. van der Hoek and B. van Linder, *A logical approach to the dynamics of commitments*, Artifitial Intellegence **113** (1999), pp. 1–40.

[15] Michie, D., *Machine intelligence at edinburgh*, in: *On Machine Intelligence*, 1974 pp. 143–155.

[16] Moore, R., *A formal theory of knowledge and action*, in: J. Hobbs and R. Moore, editors, *Formal Theories of the Commonsense World* (1985), pp. 319–358.

[17] Panangaden, P., C. Phillips, D. Precup and M. Sadrzadeh, *An algebraic approach to dynamic epistemic logic*, in: *Proceeedings of Description Logics'10*, 2010.

[18] Panangaden, P. and M. Sadrzadeh, *Learning in a changing world, an algebraic modal logical approach*, in: *AMAST'10*, 2010, pp. 128–141.

[19] Parikh, R. and R. Ramanujam, *Distributed processes and the logic of knowledge*, in: *Proceedings of Conference on Logic of Programs* (1985), pp. 256–268.

[20] Plaza, J. A., *Logics of public communications*, in: M. L. Emrich, M. S. Pfeifer, M. Hadzikadic and Z. W. Ras, editors, *Proceedings of the 4th International Symposium on Methodologies for Intelligent Systems*, 1989, pp. 201–216.

[21] van Benthem, J., J. Gerbrandy, T. Hoshi and E. Pacuit, *Merging frameworks for interaction*, Journal of Philosophical Logic **38** (2009), pp. 491–526.

[22] van der Hoek, W., N. Troquard and M. Wooldridge, *Knowledge and control*, in: S. Tumer, Yolum and Stone, editors, *Proceedings of AAMAS'11*, 2011, pp. 719–726.

[23] van Ditmarsch, H., W. van der Hoek and B. Kooi, *Dynamic epistemic logic with assignment*, in: *Proceedings of AAMAS'05*, 2005, pp. 141–148.

[24] van Ditmarsch, H., W. van der Hoek and B. Kooi, "Dynamic Epistemic Logic," (Synthese Library), Springer, 2007, 1st edition.

[25] Wang, Y. and Q. Cao, *On axiomatizations of public announcement logic* (2012), manuscript under submission.