THE REVIEW OF SYMBOLIC LOGIC
Volume 8, Number 1, March 2015

CONTINGENCY AND KNOWING WHETHER

JIE FAN, YANJING WANG
Department of Philosophy, Peking University

and

HANS VAN DITMARSCH
LORIA-CNRS/Univeristy of Lorraine

Abstract. A proposition is noncontingent, if it is necessarily true or it is necessarily false. In an epistemic context, 'a proposition is noncontingent' means that you know *whether* the proposition is true. In this paper, we study contingency logic with the noncontingency operator Δ but without the necessity operator $\Box$. This logic is not a normal modal logic, because $\Delta(\varphi \to \psi) \to (\Delta\varphi \to \Delta\psi)$ is not valid. Contingency logic cannot define many usual frame properties, and its expressive power is weaker than that of basic modal logic over classes of models without reflexivity. These features make axiomatizing contingency logics nontrivial, especially for the axiomatization over *symmetric* frames. In this paper, we axiomatize contingency logics over various frame classes using a novel method other than the methods provided in the literature, based on the 'almost-definability' schema AD proposed in our previous work. We also present extensions of contingency logic with dynamic operators. Finally, we compare our work to the related work in the fields of contingency logic and ignorance logic, where the two research communities have similar results but are apparently unaware of each other's work. One goal of our paper is to bridge this gap.

§1. Introduction.

A proposition is contingent if it is possibly true and it is possibly false. A proposition is noncontingent, if it is not contingent, i.e., if it is necessarily true or it is necessarily false. In a *doxastic* context, 'a proposition is contingent' means that you are agnostic about the value of the proposition, while 'a proposition is noncontingent' means that you are opinionated as to whether the proposition is true. In an *epistemic* context, 'a proposition is contingent' means that you are ignorant about the truth value of the proposition, while 'a proposition is noncontingent' means that you *know whether* the proposition is true.

In the epistemic setting, 'knowing whether φ' is a very natural and succinct statement which is often sufficient to express interesting propositions without using the more expressive 'knowing that' construction. For example, the 'knowing whether' operator is used frequently in problem specifications in AI to express preconditions for robots to move (McCarthy, 1979; Reiter, 2001; Petrick & Bacchus, 2004); it also facilitates a neat construction to establish a continuum of knowledge states in microeconomics (Hart *et al.*, 1996; Heifetz & Samet, 1993); moreover, intricate higher-order epistemic reasoning in scenarios such as Muddy Children (Moses *et al.*, 1986) and Gossip Protocols (Hedetniemi

Received: December 02, 2013.

doi:10.1017/S1755020314000343

et al., 1988) can be formalized in terms of 'knowing whether'.[1] Besides discussions of 'knowing whether' in an epistemic logic with alternative questions such as Aloni *et al.* (2013) and in inquisitive semantics such as Ciardelli & Roelofsen (2014), there are also previous works trying to axiomatize the *logic of ignorance*, where the negated operator of 'knowing whether' is considered (van der Hoek & Lomuscio, 2003, 2004; Steinsvold, 2008). An axiomatization of the logic of ignorance over the class of arbitrary frames is given in van der Hoek & Lomuscio (2003, 2004). The authors suggest that it is hard to repeat this exercise for other frame classes. Moreover, 'knowing whether' can also be viewed as a special case of 'knowing value': knowing whether φ is knowing the truth value of φ (cf. Plaza, 1989; van Ditmarsch, 2007; Wang & Fan, 2013, 2014).

In a nonepistemic setting, 'knowing whether' can be read as *noncontingency*. Contingency is an important concept in philosophy and philosophical logic, which dates back to Aristotle (cf. e.g., Brogan, 1967). Montgomery & Routley (1966) first define contingency in modal logic: a proposition φ is *contingent*, if it is possibly true and it is possibly false; otherwise, φ is *noncontingent*. One main theme in the logic literature of contingency, is axiomatizing contingency logic, i.e. the logic with contingency operator as the sole modality. Unlike standard modal logic, contingency logic cannot define the usual frame properties. This makes it nontrivial to find the axiomatizations of contingency logic over different frame classes. An unpublished axiomatization for contingency-based **S5** was proposed by Lemmon and Gjertsen in 1959 (Humberstone, 2002, note 10). The logics of contingency-based **T**, **S4** and **S5** are axiomatized in Montgomery & Routley (1966). Humberstone (1995) provides an infinite axiomatization for contingency logic over **K**-frames and over **D**-frames. A finite axiomatization is proposed by Kuhn (1995), which also gives a finite axiomatization for transitive contingency logic. Euclidean contingency logic is axiomatized by Zolin (1999).

To our surprise, the people working on contingency logic and the people working on the logic of ignorance are unaware of each other's work. We hope our paper can bridge the gap.

Although various axiomatizations scattered in the epistemic and nonepistemic literature, there has been no uniform method for completely axiomatizing contingency logic over the usual frame classes, and this motivates the current paper. In this paper, based on the *almost-definability* schema (AD) that we proposed earlier in Fan *et al.* (2014), we use a highly uniform method which is different from the ones in the literature to show the completeness of contingency logics over various classes of frames. In particular, our method applies to the multimodal contingency logic which makes perfect sense in the multi-agent epistemic setting. Interestingly, the multimodal logic may introduce technical difficulties to the completeness proof, as demonstrated by our highly nontrivial completeness proof of multimodal contingency logic over symmetric frames, which also answers an open question raised in Fan *et al.* (2014). We will compare our proof method and axiomatizations with the known ones in the literature in Section §9. Moreover, we also extend the contingency logic with dynamic operators in line with Plaza (1989) and Baltag *et al.* (1998) to handle information changes.

The table below is an overview of the known axiomatizations in the literature and the results in this paper. The first column lists the original and dynamified contingency logics,

[1] E.g., in Muddy Children, iterated truthful announcements of 'nobody knows whether he is muddy' will eventually let each muddy child know whether he is muddy. Epistemic versions of gossip protocols using 'knowing whether' are treated in Attamah *et al.* (2014).

while the former ranges over various frame classes, the latter includes public announcements and action models as additional modalities to the language. The second column lists the known axiomatizations (in the unimodal case), and the third column concerns our new axiomatizations/completeness proofs of the multimodal case.

Frame classes	Known axiomatizations (unimodal)	Our results (multimodal)
$\mathcal{K}$	Humberstone (1995), Kuhn (1995), and Zolin (1999); van der Hoek & Lomuscio (2004)	Thm. 4.7
$\mathcal{D}$	Humberstone (1995) and Zolin (1999)	Thm. 5.6
$\mathcal{T}$	Montgomery & Routley (1966)	Thm. 5.9
4	Kuhn (1995) and Zolin (1999)	Thm. 5.10
5	Zolin (1999)	Thm. 5.11
$\mathcal{B}$	Fan *et al.* (2014)	Thm. 6.13
$\mathcal{KD}$45	Zolin (1999)	Thm. 5.13
$\mathcal{S}$4	Montgomery & Routley (1966) and Steinsvold (2008)	Thm. 5.14
$\mathcal{S}$5	Montgomery & Routley (1966)	Thm. 5.15
Public Announcements	—	Thm. 7.5
Action Models	—	Thm. 8.6

The rest of the paper is structured as follows. In Section §2 we define the language and semantics of contingency logic, and present an almost-definability schema which is the multimodal version of the one in Fan *et al.* (2014). Section §3 deals with expressivity over models and with frame correspondence, and Section §4 presents our new axiomatization of the logic over the class of arbitrary frames and proves its completeness. Then, in Section §5 we give axiomatizations for other frame classes. As one of the key results, we axiomatize the multimodal contingency logic over *symmetric* frames in Section §6, thereby answering an open question raised in Fan *et al.* (2014). We also consider dynamic contingency logics: contingency logic with public announcements in Section §7 and contingency logic with action models in Section §8. The above-mentioned muddy children problem and gossip protocols illustrate these two logics. In Section §9 we compare our axiomatizations and proof method with the ones in the literature on ignorance logic and contingency logic. We conclude and list some further directions in Section §10.

§2. Syntax and semantics of contingency logic. We first define a (multimodal) logical language including both noncontingency and necessity operators. The standard modal language and the language of contingency logic can be viewed as two fragments of this language and we will mainly focus on the latter in the rest of the paper.

DEFINITION 2.1 (Logical languages **CML**, **CL** and **ML**). *Let a set* $\boldsymbol{P}$ *of propositional variables and a finite set* $\boldsymbol{I}$ *be given.*[2] *The logical language* $\boldsymbol{CML}(\boldsymbol{P}, \boldsymbol{I})$ *is defined as:*

$$\varphi ::= \top \mid p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Delta_i \varphi \mid \Box_i \varphi$$

[2] The assumption that "**I** is finite" is crucial in the completeness proof of system $\mathbb{CLB}$ in Section §6.

where $p \in \mathbf{P}$ and $i \in \mathbf{I}$. Without the $\Box_i\varphi$ construct, we have the language $\mathbf{CL(P, I)}$ of contingency logic. *Without the $\Delta_i\varphi$ construct, we have the* language $\mathbf{ML(P, I)}$ of modal logic.

We typically omit the parameters **P** and **I** from the notations for these languages. Intuitively, we can view **I** as a set of *agents* who may have different opinions on the necessity and contingency of propositions. Thus the formula $\Box_i\varphi$ stands for 'φ is necessary for agent i', and the formula $\Delta_i\varphi$ stands for 'φ is noncontingent for agent i', namely, for i, φ is necessarily true or φ is necessarily false. In a doxastic context ($\mathcal{KD}45$), $\Box_i\varphi$ and $\Delta_i\varphi$ mean, respectively, that 'agent i believes that φ' and 'i is *opinionated* as to whether φ'. In an epistemic context ($\mathcal{S}5$), $\Box_i\varphi$ and $\Delta_i\varphi$ mean, respectively, that 'agent i knows that φ' and 'agent i knows whether φ' (i.e., i knows that φ is true or i knows that φ is false), although we do not restrict ourselves to epistemic or doxastic contexts. As usual, we define $\bot$, $(\varphi \vee \psi)$, $(\varphi \to \psi)$, $(\varphi \leftrightarrow \psi)$, $\nabla_i\varphi$ as the abbreviations of, respectively, $\neg\top$, $\neg(\neg\varphi \wedge \neg\psi)$, $(\neg\varphi \vee \psi)$, $((\varphi \to \psi) \wedge (\psi \to \varphi))$ and $\neg\Delta_i\varphi$. Note that $\nabla_i\varphi$ is *not* defined as the dual but the negation of $\Delta_i\varphi$, which expresses 'φ is contingent for i'.[3] We omit parentheses from formulas unless confusion results. In particular, we assume that $\wedge$ and $\vee$ bind stronger than $\to$ and $\leftrightarrow$. For $\varphi_1 \wedge \cdots \wedge \varphi_m$ we write $\bigwedge_{j=1}^m \varphi_j$, and for $\varphi_1 \vee \cdots \vee \varphi_m$ we write $\bigvee_{j=1}^m \varphi_j$.

DEFINITION 2.2 (Model). *A* model *is a triple $\mathcal{M} = \langle S, \{\to_i \mid i \in \mathbf{I}\}, V\rangle$ where S is a nonempty set of possible worlds, $\to_i$ is a binary relation over S for each $i \in \mathbf{I}$, and V is a valuation function assigning a set of worlds $V(p) \subseteq S$ to each $p \in \mathbf{P}$. Given a world $s \in S$, the pair $(\mathcal{M}, s)$ is a* pointed model. *A* frame *is a pair $\mathcal{F} = \langle S, \{\to_i \mid i \in \mathbf{I}\}\rangle$, i.e., a model without a valuation. We will refer to special classes of models or frames using the notation below.*

Notation	Frame Property
$\mathcal{K}$	—
$\mathcal{D}$	*seriality*
$\mathcal{T}$	*reflexivity*
$\mathcal{B}$	*symmetry*
4	*transitivity*
5	*Euclidicity*
45	*transitivity, Euclidicity*
$\mathcal{KD}45$	*seriality, transitivity, Euclidicity*
$\mathcal{S}4$	*reflexivity, transitivity*
$\mathcal{S}5$	*reflexivity, Euclidicity*
$\mathcal{PF}$	*partial functionality*

where a binary relation is partial-functional *if it corresponds to a partial function, i.e., every world has at most one i-successor for each i.*

We will omit parenthesis around pointed models $(\mathcal{M}, s)$ whenever convenient. The nonstandard notion of partial functionality plays a special role in contingency logic.

[3] However, we will see that $\neg\Delta_i\varphi$ is equivalent to $\neg\Delta_i\neg\varphi$ based on the semantics.

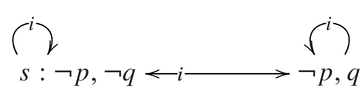
DEFINITION 2.3 (Semantics). *Given a model $\mathcal{M} = \langle S, \{\to_i \mid i \in \boldsymbol{I}\}, V\rangle$, the semantics of **CML** is defined as follows:*

$\mathcal{M}, s \vDash \top$		$always$
$\mathcal{M}, s \vDash p$	$\Leftrightarrow$	$s \in V(p)$
$\mathcal{M}, s \vDash \neg\varphi$	$\Leftrightarrow$	$\mathcal{M}, s \nvDash \varphi$
$\mathcal{M}, s \vDash \varphi \wedge \psi$	$\Leftrightarrow$	$\mathcal{M}, s \vDash \varphi$ *and* $\mathcal{M}, s \vDash \psi$
$\mathcal{M}, s \vDash \Delta_i\varphi$	$\Leftrightarrow$	*for all* t_1, t_2 *such that* $s \to_i t_1, s \to_i t_2$: $(\mathcal{M}, t_1 \vDash \varphi \Leftrightarrow \mathcal{M}, t_2 \vDash \varphi)$
$\mathcal{M}, s \vDash \Box_i\varphi$	$\Leftrightarrow$	*for all* t *such that* $s \to_i t : \mathcal{M}, t \vDash \varphi$

If $\mathcal{M}, s \vDash \varphi$ we say that φ is true *in $(\mathcal{M}, s)$, and sometimes write $s \vDash \varphi$ if $\mathcal{M}$ is clear; if for all s in $\mathcal{M}$ we have $\mathcal{M}, s \vDash \varphi$ we say that φ is* valid on $\mathcal{M}$ *and write $\mathcal{M} \vDash \varphi$; if for all $\mathcal{M}$ based on $\mathcal{F}$ with $\mathcal{M} \vDash \varphi$ we say that φ is* valid on $\mathcal{F}$ *and write $\mathcal{F} \vDash \varphi$; if for all $\mathcal{F}$ with $\mathcal{F} \vDash \varphi$, φ is* valid *and we write $\vDash \varphi$. Given $\Phi \subseteq$ **CML**, $\mathcal{M}, s \models \Phi$ stands for 'for all $\varphi \in \Phi$, $\mathcal{M}, s \models \varphi$,' and similarly for model/frame validity, and validity. If there exists an $(\mathcal{M}, s)$ such that $\mathcal{M}, s \vDash \varphi$, then φ is* satisfiable.

Intuitively, $\Delta_i\varphi$ is true at s if and only if φ has the same truth value on the worlds that i thinks possible at s. Contingency logic is *not* normal, because $\Delta_i(\varphi \to \psi) \to (\Delta_i\varphi \to \Delta_i\psi)$ is invalid (and, in relation to that, $\vDash \varphi \to \psi$ does not imply $\vDash \Delta_i\varphi \to \Delta_i\psi$). In the **S5**-model $\mathcal{M}_1$ below we have that $\mathcal{M}_1, s \vDash \Delta_i(p \to q)$ and $\mathcal{M}_1, s \vDash \Delta_i p$, but $\mathcal{M}_1, s \nvDash \Delta_i q$.

$\mathcal{M}_1$:



We use $\varphi[\psi/p]$ to denote a *uniform substitution* of φ, i.e., the formula obtained by replacing all occurrences of p in φ (if there is any) with ψ. It can be shown that uniform substitution preserves the validity of **CL**-formulas.

PROPOSITION 2.4. *For any $\psi, \varphi \in$ **CL**, any $p \in \boldsymbol{P}$: if $\vDash \varphi$, then $\vDash \varphi[\psi/p]$.*

§3. Expressivity and frame correspondence. In this section we compare the relative expressivity of contingency logic and modal logic, and we give some negative results for frame correspondence for contingency logic.

3.1. Expressivity. We adopt the definition of expressivity in van Ditmarsch *et al.* (2007, Def. 8.2).

DEFINITION 3.1 (Expressive). *Given two logical languages L_1 and L_2 that are interpreted in the same class $\mathbb{M}$ of models,*

- *L_2 is* at least as expressive as *L_1, notation $L_1 \preceq L_2$, if and only if for every formula $\varphi_1 \in L_1$ there is a formula $\varphi_2 \in L_2$ such that for all $(\mathcal{M}, s) \in \mathbb{M}$, $\mathcal{M}, s \vDash \varphi_1$ iff $\mathcal{M}, s \vDash \varphi_2$.*
- *L_1 and L_2 are* equally expressive*, notation $L_1 \equiv L_2$, if and only if $L_1 \preceq L_2$ and $L_2 \preceq L_1$.*
- *L_1 is* less expressive than *L_2, notation $L_1 \prec L_2$, if and only if $L_1 \preceq L_2$ and $L_2 \npreceq L_1$.*

PROPOSITION 3.2. ***CL** is less expressive than **ML** on the class of $\mathcal{K}$-models, $\mathcal{D}$-models, 4-models, 5-models.*

Proof. This is a truth-preserving translation t from **CL** to **ML**:

$$
\begin{array}{lcl}
t(p) & = & p \\
t(\neg\varphi) & = & \neg t(\varphi) \\
t(\varphi \wedge \psi) & = & t(\varphi) \wedge t(\psi) \\
t(\Delta_i \varphi) & = & \Box_i t(\varphi) \vee \Box_i \neg t(\varphi)
\end{array}
$$

Therefore **ML** is at least as expressive as **CL**. But **CL** is not at least as expressive as **ML**: even the simplest **ML** formula $\Box_i p$ does not have an equivalent **CL** correspondent. The pointed models $(\mathcal{M}, s)$ and $(\mathcal{N}, t)$ below, which are distinguished by $\Box_i p$, cannot be distinguished by a **CL** formula.

Note that $\mathcal{M}$ and $\mathcal{N}$ are serial, transitive, and Euclidean. By induction we prove: for any $\varphi \in$ **CL**, $\mathcal{M}, s \vDash \varphi$ iff $\mathcal{N}, t \vDash \varphi$. The nontrivial case is $\varphi = \Delta_i \psi$. Note that s and t both have only one successor. Therefore, for all ψ, $\mathcal{M}, s \nvDash \Delta_i \psi$ and $\mathcal{N}, t \nvDash \Delta_i \psi$, so also, as required, $\mathcal{M}, s \vDash \Delta_i \psi$ iff $\mathcal{N}, t \vDash \Delta_i \psi$ (note that we do not need the induction hypothesis here). $\Box$

PROPOSITION 3.3. ***CL** is less expressive than **ML** on the class of $\mathcal{B}$-models.*

Proof. Consider the following $\mathcal{B}$-models $(\mathcal{M}', s')$ and $(\mathcal{N}', t')$. Again, they are distinguished by $\Box_i p$, but are modally equivalent in **CL** (by a similar argument as in Prop. 3.2).

$\Box$

Note that the above two propositions can also be obtained by using the notion of Δ-bisimulation in Fan *et al.* (2014).

However, on the class of $\mathcal{T}$-models, **CL** and **ML** are equally expressive.

PROPOSITION 3.4 (Demri, 1997). ***CL** and **ML** are equally expressive on the class of $\mathcal{T}$-models.*

Proof. Consider translation $t' : \textbf{ML} \to \textbf{CL}$:

$$
\begin{array}{lcl}
t'(p) & = & p \\
t'(\neg\varphi) & = & \neg t'(\varphi) \\
t'(\varphi \wedge \psi) & = & t'(\varphi) \wedge t'(\psi) \\
t'(\Box_i \varphi) & = & t'(\varphi) \wedge \Delta_i t'(\varphi)
\end{array}
$$

On the class of $\mathcal{T}$-models, t' is truth preserving (elementary, by induction on φ in $t'(\varphi)$). This demonstrates that **ML** $\preceq$ **CL**. As we already had **CL** $\preceq$ **ML**, by way of translation t defined in the proof of Prop. 3.2, we get that **ML** $\equiv$ **CL** on $\mathcal{T}$-models. $\Box$

This result applies to any model class contained in $\mathcal{T}$, such as $\mathcal{S}4$ and $\mathcal{S}5$.

We close this section on expressivity with a curious observation related to (although not strictly about) expressivity. We now know that necessity *cannot* be defined in terms of contingency on $\mathcal{K}$, but that necessity *can* be defined in terms of contingency on $\mathcal{T}$. It is therefore interesting to observe that under slightly stronger conditions, necessity can still be 'defined' in terms of contingency on $\mathcal{K}$, namely, given a model, in a world of that model wherein some proposition is contingent for the agent. We call this 'almost-definability schema'. Roughly, it says that a proposition is necessary, if and only if it is noncontingent, and it is noncontingently implied by a contingent proposition. We refer the reader to Fan *et al.* (2014, Prop. 2.5) for the proof details, where the unimodal case was proved.

PROPOSITION 3.5. *Let* $\varphi, \chi \in \boldsymbol{CL}$ *and* $i \in \boldsymbol{I}$. Almost-definability *is the schema AD*

$$\nabla_i \chi \to (\Box_i \varphi \leftrightarrow \Delta_i \varphi \wedge \Delta_i(\chi \to \varphi)).$$

Almost-definability AD is a validity of ***CML****.*

The almost-definability schema is very important. It motivates the canonical relation in the construction of canonical model for contingency logics, as we will see in Section §4. With this relation or some adaption we can show the completeness of all axiomatizations mentioned in the introduction *uniformly*.

3.2. Frame correspondence. Standard modal logic formulas can be used to capture frame properties, e.g., $\Box p \to p$ corresponds to the reflexivity of frames. It is therefore remarkable that in contingency logic there is no such correspondence for most of the basic frame properties. In this section we show the undefinability results with the method *much simpler* than that used in Zolin (1999), where the author needs to show a complicated theorem to the effect that every contingency-definable class of frames contains the class of partial-functional frames. For the definition of partial functionality, see Def. 2.2.

DEFINITION 3.6 (Frame definability). *Let* Φ *be a set of* ***CL****-formulas and* F *a class of frames. We say that* Φ *defines* F *if for all frames* $\mathcal{F}$, $\mathcal{F}$ *is in* F *if and only if* $\mathcal{F} \vDash \Phi$. *In this case we also say* Φ *defines the property of* F. *If* Φ *is a singleton (e.g.* $\{\varphi\}$*), we usually write* $\mathcal{F} \vDash \varphi$ *rather than* $\mathcal{F} \vDash \{\varphi\}$. *A class of frames (or the corresponding frame property) is definable in* ***CL*** *if there is a set of* ***CL****-formulas that defines it.*

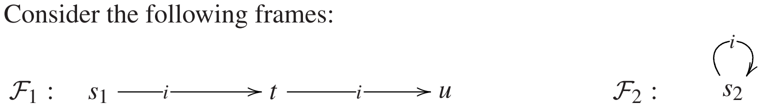
PROPOSITION 3.7. *For any partial-functional frames* $\mathcal{F}, \mathcal{F}'$ *and any* $\varphi \in \boldsymbol{CL}$*:* $\mathcal{F} \vDash \varphi$ *iff* $\mathcal{F}' \vDash \varphi$.

Proof. Let $\mathcal{F} = \langle S, \{\to_i \mid i \in \mathbf{I}\}\rangle$ and $\mathcal{F}' = \langle S', \{\to'_i \mid i \in \mathbf{I}\}\rangle$ be two partial-functional frames, and let $\varphi \in \mathbf{CL}$.

Suppose that $\mathcal{F} \nvDash \varphi$, then there exists $\mathcal{M} = \langle \mathcal{F}, V\rangle$ and $s \in S$ such that $\mathcal{M}, s \nvDash \varphi$. Since $S' \neq \emptyset$, we may assume that $s' \in S'$. Define a valuation V' on $\mathcal{F}'$ as $p \in V'(s')$ iff $p \in V(s)$ for all $p \in \mathbf{P}$. Since $\mathcal{F}$ and $\mathcal{F}'$ are both partial-functional, both s and s' has at most one successor. By induction on $\psi \in \mathbf{CL}$, we can show that $\mathcal{M}, s \vDash \psi$ iff $\mathcal{M}', s' \vDash \psi$. From this and $\mathcal{M}, s \nvDash \varphi$, it follows that $\mathcal{M}', s' \nvDash \varphi$, therefore $\mathcal{F}' \nvDash \varphi$. The converse is similar. $\Box$

PROPOSITION 3.8 (Zolin, 1999). *The frame properties of seriality, reflexivity, transitivity, symmetry, and Euclidicity are not definable in* ***CL****.*

Proof. Consider the following frames:



Both frames are partial-functional. Thus we have: for any $\Phi \subseteq \mathbf{CL}$, $\mathcal{F}_1 \vDash \Phi$ iff $\mathcal{F}_2 \vDash \Phi$. Now observe that $\mathcal{F}_2$ is reflexive (resp. serial, transitive, symmetric, Euclidean) but $\mathcal{F}_1$ is not.

The argument now goes as follows. Consider reflexivity: If Φ were to define reflexivity, then, as $\mathcal{F}_2$ is reflexive, we have $\mathcal{F}_2 \vDash \Phi$. But as $\mathcal{F}_2$ and $\mathcal{F}_1$ satisfy the same frame validities, we also have that $\mathcal{F}_1 \vDash \Phi$. However, $\mathcal{F}_1$ is not reflexive. Therefore such a Φ does not exist. Therefore, reflexivity is not frame definable in **CL**.

The argument is similar for the other cases. (Observe that $\mathcal{F}_1$ is indeed not Euclidean, because $s_1 \rightarrow_i t$ and $s_1 \rightarrow_i t$, but it is not the case that $t \rightarrow_i t$.) □

As a consequence of this result, the axiomatizations of contingency logic over special frame classes, such as the class of reflexive frames, cannot be shown by the standard method of adding the corresponding frame axioms to the axiomatization of **CL**. This will be addressed in Section §5.

§4. Axiomatization. In this section we give a complete Hilbert-style proof system for the logic **CL** on the class of all frames. We will compare our axiomatizations and the completeness proof method to those in the literature in Section §9.

4.1. Proof system and soundness.

DEFINITION 4.1 (Proof system $\mathbb{CL}$). *The proof system $\mathbb{CL}$ consists of the following axiom schemas and inference rules.*

TAUT	*all instances of tautologies*
ΔCon	$\Delta_i(\chi \to \varphi) \wedge \Delta_i(\neg\chi \to \varphi) \to \Delta_i\varphi$
ΔDis	$\Delta_i\varphi \to \Delta_i(\varphi \to \psi) \vee \Delta_i(\neg\varphi \to \chi)$
$\Delta\leftrightarrow$	$\Delta_i\varphi \leftrightarrow \Delta_i\neg\varphi$
MP	*From* φ *and* $\varphi \to \psi$ *infer* ψ
NECΔ	*From* φ *infer* $\Delta_i\varphi$
REΔ	*From* $\varphi \leftrightarrow \psi$ *infer* $\Delta_i\varphi \leftrightarrow \Delta_i\psi$

A derivation *of $\mathbb{CL}$ is a finite sequence of **CL**-formulas such that each formula is either the instantiation of an axiom or the result of applying an inference rule to prior formulas in the sequence. A formula $\varphi \in$ **CL** is called* provable*, or a* theorem*, notation $\vdash \varphi$, if it occurs in a derivation of $\mathbb{CL}$.*

Intuitively, ΔCon means for agent i, if a formula is noncontingently implied not only by some formula but by its negation, then this formula is noncontingent; ΔDis means for agent i, if a formula is noncontingent, then either this formula is necessary, in which case its negation noncontingently implies any formula, or it is impossible, in which case it noncontingently implies any formula; $\Delta\leftrightarrow$ means for agent i, a formula is noncontingent is the same as its negation is noncontingent.

Note that the rule NECΔ is *not* admissible in the system $\mathbb{CL}$ − NECΔ, which means NECΔ is indispensable in $\mathbb{CL}$.[4] To see this, we can show that $\Delta_i\top$ is not provable in $\mathbb{CL}$ − NECΔ: define an auxiliary semantics $\Vdash$, which is the same as $\vDash$ except that wherein each $\Delta_i\varphi$ is interpreted as *false*. Then we can show that $\mathbb{CL}$ − NECΔ is sound with respect to $\Vdash$, but $\nVdash \Delta_i\top$, thus $\Delta_i\top$ is not provable in $\mathbb{CL}$ − NECΔ, therefore NECΔ is not admissible in $\mathbb{CL}$ − NECΔ.

PROPOSITION 4.2. *The proof system $\mathbb{CL}$ is sound with respect to the class of all frames.*

[4] This claim also applies to systems $\mathbb{CLT}$, $\mathbb{CL}4$ and $\mathbb{CLS}4$ in this paper (see Def. 5.1), for which the proof is similar.

Proof. The soundness of $\mathbb{CL}$ follows immediately from the validity of three crucial axioms. The other axioms and the derivation rules are obviously valid. We prove that: for any $\varphi, \psi, \chi \in \mathbf{CL}$ and any $i \in \mathbf{I}$,

1. $\Delta\texttt{Con}$ is valid: $\vDash \Delta_i(\chi \to \varphi) \wedge \Delta_i(\neg\chi \to \varphi) \to \Delta_i\varphi$
2. $\Delta\texttt{Dis}$ is valid: $\vDash \Delta_i\varphi \to \Delta_i(\varphi \to \psi) \vee \Delta_i(\neg\varphi \to \chi)$
3. $\Delta\leftrightarrow$ is valid: $\vDash \Delta_i\varphi \leftrightarrow \Delta_i\neg\varphi$

3 is immediate from the semantics of Δ_i.

For 1, assume towards a contradiction that for some $(\mathcal{M}, s)$ such that $\mathcal{M}, s \vDash \Delta_i(\chi \to \varphi)$, $\mathcal{M}, s \vDash \Delta_i(\neg\chi \to \varphi)$ but $\mathcal{M}, s \vDash \neg\Delta_i\varphi$, then there exist t_1, t_2 such that $s \to_i t_1, s \to_i t_2$ and $t_1 \vDash \varphi, t_2 \vDash \neg\varphi$. Clearly, with $t_1 \vDash \varphi$ we get $t_1 \vDash \chi \to \varphi$ and $t_1 \vDash \neg\chi \to \varphi$. Thus from the fact that $s \vDash \Delta_i(\chi \to \varphi), s \to_i t_1, s \to_i t_2$ and $t_1 \vDash \chi \to \varphi$ we get $t_2 \vDash \chi \to \varphi$. Similarly, by using $t_1 \vDash \neg\chi \to \varphi$ we can get $t_2 \vDash \neg\chi \to \varphi$. Now we obtain $t_2 \vDash \chi \to \varphi$ and $t_2 \vDash \neg\chi \to \varphi$, therefore $t_2 \vDash \varphi$. Contradiction.

For 2, let $(\mathcal{M}, s)$ be an arbitrary model. Suppose via contraposition that $\mathcal{M}, s \vDash \neg\Delta_i(\varphi \to \psi)$ and $\mathcal{M}, s \vDash \neg\Delta_i(\neg\varphi \to \chi)$, we only need to show $\mathcal{M}, s \vDash \neg\Delta_i\varphi$. By supposition, there exist t_1, t_2 such that $s \to_i t_1, s \to_i t_2$ and $t_1 \vDash \varphi \to \psi, t_2 \vDash \neg(\varphi \to \psi)$ and, there exist u_1, u_2 such that $s \to_i u_1, s \to_i u_2$ and $u_1 \vDash \neg\varphi \to \chi, u_2 \vDash \neg(\neg\varphi \to \chi)$, respectively. From $t_2 \vDash \neg(\varphi \to \psi)$ and $u_2 \vDash \neg(\neg\varphi \to \chi)$ it follows $t_2 \vDash \varphi$ and $u_2 \vDash \neg\varphi$ respectively. So far we have shown $s \to_i t_2, s \to_i u_2$ and $t_2 \vDash \varphi, u_2 \vDash \neg\varphi$, therefore we conclude that $\mathcal{M}, s \vDash \neg\Delta_i\varphi$, as desired. □

Using the rule $\texttt{RE}\Delta$, by induction on χ we can show

PROPOSITION 4.3. *Consider the inference rule* Substitution of equivalents*:*

$$\texttt{Sub} \quad \textit{From } \varphi \leftrightarrow \psi, \ \textit{infer } \chi[\varphi/p] \leftrightarrow \chi[\psi/p]$$

Substitution of equivalents is admissible in $\mathbb{CL}$*.*

The inference rule $\texttt{RE}\Delta$ in the system $\mathbb{CL}$ is crucial. Consider again the schema

$$\texttt{K} \quad \Delta_i(\varphi \to \psi) \to (\Delta_i\varphi \to \Delta_i\psi)$$

We have already shown in Section §2 that K is invalid. This axiom is typically used to prove Sub, but is lacking in $\mathbb{CL}$. Without $\texttt{RE}\Delta$, Sub is not admissible in $\mathbb{CL}$.

We first prove a proposition, which is the multimodal version of Fan *et al.* (2014, Prop. 5.5) (the proofs are different). It will be used in Lemma 4.6. Intuitively, it says that for agent i, if a formula is noncontingently implied by a conjunction of which each conjunct is noncontingent, and the negation of the formula noncontingently implies all conjuncts, then the formula is itself noncontingent.

PROPOSITION 4.4. *For all* $k \geq 1$*:*

$$\vdash \Delta_i(\bigwedge_{j=1}^{k} \varphi_j \to \neg\psi) \wedge \bigwedge_{j=1}^{k} \Delta_i\varphi_j \wedge \bigwedge_{j=1}^{k} \Delta_i(\psi \to \varphi_j) \to \Delta_i\psi$$

Proof. By induction on k.

- Base step. We need to show that $\vdash \Delta_i(\varphi_1 \to \neg\psi) \wedge \Delta_i\varphi_1 \wedge \Delta_i(\psi \to \varphi_1) \to \Delta_i\psi$. This is clear from TAUT, $\texttt{RE}\Delta$, $\Delta\texttt{Con}$ and $\Delta\leftrightarrow$.

- Inductive step. Assume by induction hypothesis (IH) that the proposition holds for $k = n$. We now need to show that:

$$\vdash \Delta_i(\bigwedge_{j=1}^{n+1} \varphi_j \to \neg\psi) \wedge \bigwedge_{j=1}^{n+1} \Delta_i\varphi_j \wedge \bigwedge_{j=1}^{n+1} \Delta_i(\psi \to \varphi_j) \to \Delta_i\psi$$

The proof is as follows.

(i)	$\Delta_i(\neg\varphi_{n+1} \to \neg\psi) \wedge \Delta_i(\varphi_{n+1} \to \neg\psi) \to \Delta_i\neg\psi$	ΔCon
(ii)	$\Delta_i(\psi \to \varphi_{n+1}) \wedge \neg\Delta_i\psi \to \neg\Delta_i(\varphi_{n+1} \to \neg\psi)$	TAUT, REΔ, $\Delta\leftrightarrow$, (i)
(iii)	$\Delta_i\varphi_{n+1} \to \Delta_i(\varphi_{n+1} \to \neg\psi)$ $\vee\Delta_i(\neg\varphi_{n+1} \to (\bigwedge_{j=1}^{n} \varphi_j \to \neg\psi))$	ΔDis
(iv)	$\Delta_i(\psi \to \varphi_{n+1}) \wedge \neg\Delta_i\psi \wedge \Delta_i\varphi_{n+1}$ $\to \Delta_i(\neg\varphi_{n+1} \to (\bigwedge_{j=1}^{n} \varphi_j \to \neg\psi))$	TAUT$(ii)(iii)$
(v)	$\Delta_i(\neg\varphi_{n+1} \to (\bigwedge_{j=1}^{n} \varphi_j \to \neg\psi))$ $\wedge\Delta_i(\varphi_{n+1} \to (\bigwedge_{j=1}^{n} \varphi_j \to \neg\psi))$ $\to \Delta_i((\bigwedge_{j=1}^{n} \varphi_j \to \neg\psi))$	ΔCon
(vi)	$\Delta_i(\psi \to \varphi_{n+1}) \wedge \neg\Delta_i\psi \wedge \Delta_i\varphi_{n+1}$ $\wedge\Delta_i(\bigwedge_{j=1}^{n+1} \varphi_j \to \neg\psi) \to \Delta_i((\bigwedge_{j=1}^{n} \varphi_j \to \neg\psi))$	TAUT, REΔ, $(iv)(v)$
(vii)	$\Delta_i(\bigwedge_{j=1}^{n} \varphi_j \to \neg\psi) \wedge \bigwedge_{j=1}^{n} \Delta_i\varphi_j$ $\wedge \bigwedge_{j=1}^{n} \Delta_i(\psi \to \varphi_j) \to \Delta_i\psi$	IH
(viii)	$\Delta_i(\bigwedge_{j=1}^{n+1} \varphi_j \to \neg\psi) \wedge \bigwedge_{j=1}^{n+1} \Delta_i\varphi_j \wedge \neg\Delta_i\psi$ $\wedge \bigwedge_{j=1}^{n+1} \Delta_i(\psi \to \varphi_j) \to \Delta_i\psi$	TAUT$(vi)(vii)$
(ix)	$\Delta_i(\bigwedge_{j=1}^{n+1} \varphi_j \to \neg\psi) \wedge \bigwedge_{j=1}^{n+1} \Delta_i\varphi_j$ $\wedge \bigwedge_{j=1}^{n+1} \Delta_i(\psi \to \varphi_j) \to \Delta_i\psi$	TAUT$(viii)$

□

4.2. Completeness. We proceed with the completeness of the proof system $\mathbb{CL}$. The completeness of the logic is shown via a canonical model construction.

DEFINITION 4.5 (Canonical model). *The canonical model $\mathcal{M}^c$ of $\mathbb{CL}$ is the tuple $\langle S^c, \{\to_i^c \mid i \in \boldsymbol{I}\}, V^c\rangle$, where:*

- $S^c = \{s \mid s$ *is a maximal consistent set of* $\mathbb{CL}\}$.
- $s \to_i^c t$ *iff there exists* χ *such that*

    1. $\neg\Delta_i\chi \in s$ *and*
    2. *for all* φ*:* $\Delta_i\varphi \wedge \Delta_i(\chi \to \varphi) \in s$ *implies* $\varphi \in t$.

- $V^c(p) = \{s \in S^c \mid p \in s\}$.

We observe that every consistent set of $\mathbb{CL}$ can be extended to a maximal consistent set of $\mathbb{CL}$ (Lindenbaum's Lemma) in the standard way. The definition of $\to_i^c$ is inspired by the almost-definability schema AD (Prop. 3.5). Recall that in the construction of canonical model for multimodal logic, the canonical relation $\to_i^c$ is usually defined by $s \to_i^c t$ iff for all φ, $\Box_i\varphi \in s$ implies $\varphi \in t$. According to the almost-definability, $\Box_i\varphi \in s$ can be replaced by $\Delta_i\varphi \wedge \Delta_i(\chi \to \varphi) \in s$ provided that $\neg\Delta_i\chi \in s$. Intuitively, if there is no $\neg\Delta_i\chi$ that holds on a world, then we do not need to add any outgoing transition.

LEMMA 4.6 (Truth Lemma). *For any **CL** formula φ, $\mathcal{M}^c, s \vDash \varphi$ iff $\varphi \in s$.*

Proof. By induction on φ. The only nontrivial case is when $\varphi = \Delta_i \psi$.

'If': Assume that $\Delta_i \psi \in s$, we need to show $\mathcal{M}^c, s \vDash \Delta_i \psi$. Suppose not, then there exist $t_1, t_2 \in S^c$ such that $s \rightarrow_i^c t_1$, $s \rightarrow_i^c t_2$ and $t_1 \vDash \psi$ and $t_2 \nvDash \psi$. From $t_1 \vDash \psi$ and $t_2 \nvDash \psi$, and induction hypothesis, we have that $\psi \in t_1$ and $\psi \notin t_2$, respectively. From $s \rightarrow_i^c t_1$ we infer that there is a χ_1 such that $\neg\Delta_i \chi_1 \in s$ and $(*)$: for all θ, $\Delta_i\theta \wedge \Delta_i(\chi_1 \to \theta) \in s$ implies $\theta \in t_1$. Since $\Delta_i \psi \in s$ and $\psi \in t_1$, $\Delta_i \neg\psi \in s$ and $\neg\psi \notin t_1$. Now from $(*)$, it follows that $\neg\Delta_i(\chi_1 \to \neg\psi) \in s$, thus $\neg\Delta_i(\psi \to \neg\chi_1) \in s$ by $\mathtt{RE}\Delta$. Similarly, from $s \rightarrow_i^c t_2$ we derive that there exists χ_2 such that $\neg\Delta_i(\chi_2 \to \psi) \in s$, i.e., $\neg\Delta_i(\neg\psi \to \neg\chi_2) \in s$. By the axiom $\Delta\mathtt{Dis}$, we obtain that $\neg\Delta_i \psi \in s$, contradiction.

'Only if': Suppose that $\Delta_i \psi \notin s$. Then $\neg\Delta_i \psi \in s$ and $\neg\Delta_i \neg\psi \in s$. We need to construct two points $t_1, t_2 \in S^c$ such that $s \rightarrow_i^c t_1$ and $s \rightarrow_i^c t_2$ and $\psi \in t_1$ and $\neg\psi \in t_2$. First, we have to show

1. $\{\varphi \mid \Delta_i\varphi \wedge \Delta_i(\psi \to \varphi) \in s\} \cup \{\psi\}$ is consistent.
2. $\{\varphi \mid \Delta_i\varphi \wedge \Delta_i(\neg\psi \to \varphi) \in s\} \cup \{\neg\psi\}$ is consistent.

We prove item 1. Suppose the set is inconsistent. Then there exist $\varphi_1, \cdots, \varphi_n$ such that $\vdash \varphi_1 \wedge \cdots \wedge \varphi_n \to \neg\psi$ and $\Delta_i\varphi_k \wedge \Delta_i(\psi \to \varphi_k) \in s$ for all $k \in [1, n]$. From $\mathtt{NEC}\Delta$ follows that $\Delta_i(\varphi_1 \wedge \cdots \wedge \varphi_n \to \neg\psi) \in s$. Now from Prop. 4.4, we infer that $\Delta_i \psi \in s$, contradiction.

From item 1, the definition of $\rightarrow_i^c$, and the observation that every consistent set can be extended to a maximal consistent set (Lindenbaum's Lemma), we conclude that there is a t_1 such that $s \rightarrow_i^c t_1$ and $\psi \in t_1$.

The proof of item 2 is similar to item 1, and similarly, from item 2, we conclude that there is a t_2 such that $s \rightarrow_i^c t_2$ and $\neg\psi \in t_2$. □

Based on Lindenbaum's Lemma and Lemma 4.6, the completeness of $\mathbb{CL}$ is immediate.

THEOREM 4.7 (Humberstone, 1995; Kuhn, 1995; Zolin, 1999). *$\mathbb{CL}$ is complete with respect to the class $\mathcal{K}$ of all frames.*[5]

Given the translation from **CL** to **ML** (i.e. the translation t in the proof of Prop. 3.2), and the decidability of **ML**, the (satisfiability problem of) contingency logic is obviously decidable.

PROPOSITION 4.8 (Decidability of **CL**). *The logic **CL** is decidable.*

§5. Axiomatization: Extensions. In this section we will give extensions of $\mathbb{CL}$ w.r.t. various classes of frames, and prove their completeness (the completeness of $\mathbb{CLB}$ will be deferred to Section §6 due to some complications in the multimodal case). Definition 5.1 shows the extra axiom schemas and corresponding systems, with on the right-hand side in the table the frame classes for which we will demonstrate completeness.

DEFINITION 5.1 (Extensions of $\mathbb{CL}$).[6]

[5] Throughout the paper, by completeness we mean strong completeness.

[6] Notice that unlike $\mathbb{CL}$, $\mathtt{NEC}\Delta$ is admissible in the system $\mathbb{CLB} - \mathtt{NEC}\Delta$, which will be shown in Prop. 6.1. This means that $\mathbb{CLB} - \mathtt{NEC}\Delta$ is already enough for axiomatizing **CL** over $\mathcal{B}$-frames. Here for uniformity, we define $\mathbb{CLB}$ as $\mathbb{CL} + \Delta\mathtt{B}$ rather than $\mathbb{CL} - \mathtt{NEC}\Delta + \Delta\mathtt{B}$.

Notation	*Axiom Schemas*	*Systems*	*Frames*
		$\mathbb{CL}$	$\mathcal{D}$
ΔT	$\Delta_i\varphi \wedge \Delta_i(\varphi \to \psi) \wedge \varphi \to \Delta_i\psi$	$\mathbb{CLT}=\mathbb{CL}+\Delta T$	$\mathcal{T}$
$\Delta 4$	$\Delta_i\varphi \to \Delta_i(\Delta_i\varphi \vee \psi)$	$\mathbb{CL}4=\mathbb{CL}+\Delta 4$	4
$\Delta 5$	$\neg\Delta_i\varphi \to \Delta_i(\neg\Delta_i\varphi \vee \psi)$	$\mathbb{CL}5=\mathbb{CL}+\Delta 5$	5
ΔB	$\varphi \to \Delta_i((\Delta_i\varphi \wedge \Delta_i(\varphi \to \psi) \wedge\neg\Delta_i\psi) \to \chi)$	$\mathbb{CLB}=\mathbb{CL}+\Delta B$	$\mathcal{B}$
$w\Delta 4$	$\Delta_i\varphi \to \Delta_i\Delta_i\varphi$	$\mathbb{CLS}4=\mathbb{CL}+\Delta T+w\Delta 4$	$\mathcal{S}4$
$w\Delta 5$	$\neg\Delta_i\varphi \to \Delta_i\neg\Delta_i\varphi$	$\mathbb{CLS}5=\mathbb{CL}+\Delta T+w\Delta 5$	$\mathcal{S}5$
		$\mathbb{CL}45=\mathbb{CL}+\Delta 4+\Delta 5$	$45\&\mathcal{K}D45$

The above axioms are found to satisfy the need, but get as close as possible to the 'translation' of the standard modal logic axioms, with the help of AD. Take Axiom ΔT for example.

$$
\begin{aligned}
& \nabla_i\neg\psi \to (\Box_i\neg\varphi \to \neg\varphi) && (1)\\
\Leftrightarrow\ & \nabla_i\neg\psi \wedge \Box_i\neg\varphi \to \neg\varphi && (2)\\
\Leftrightarrow\ & \nabla_i\neg\psi \wedge \Delta_i\neg\varphi \wedge \Delta_i(\neg\psi \to \neg\varphi) \to \neg\varphi && (3)\\
\Leftrightarrow\ & \Delta_i\varphi \wedge \Delta_i(\varphi \to \psi) \wedge \varphi \to \Delta_i\psi && (4)
\end{aligned}
$$

We write $\nabla_i\neg\psi \to (\Box_i\neg\varphi \to \neg\varphi)$ rather than $\Box_i\neg\varphi \to \neg\varphi$, since $\Box_i$ is definable in terms of Δ_i under the condition $\nabla_i\neg\psi$ for some $\neg\psi$. The above transition from (2) to (3) follows from Prop. 3.5. By using TAUT, $\Delta\leftrightarrow$, REΔ and Def ∇_i, we then get the desired axiom (4), i.e. ΔT.

It is easy to show that wΔ4 and wΔ5 are provable in $\mathbb{CL}4$ and $\mathbb{CL}5$, respectively (just let ψ in $\Delta 4$ and $\Delta 5$ be $\bot$). We will show that $\mathbb{CLS}4$ and $\mathbb{CLS}5$ are extensions of $\mathbb{CL}4$ and $\mathbb{CL}5$, respectively (Prop. 5.3). $\mathbb{CL}45$ characterizes a logic of opinionatedness, where $\Delta_i\varphi$ is read 'agent i is opinionated as to whether φ' or 'i believes φ or believes $\neg\varphi$', in a doxastic setting. Note that although 'believe whether' is not grammatical in natural language, 'believe that' is, as Égré (2008) argued. And note that we do not presuppose *neg-raising behavior* of the verb 'believe', which was assumed in Zuber (1982) and discussed in Égré (2008), so $\Delta_i\varphi$ does not hold vacuously. $\mathbb{CLS}5$ characterizes a logic of knowing whether, where $\Delta_i\varphi$ is read 'agent i knows whether φ', or 'i knows that φ or i knows that $\neg\varphi$', in an epistemic setting.

To prove the soundness of the proof systems in the above table, we only need to show:

PROPOSITION 5.2.

- *ΔT is valid on the class of all $\mathcal{T}$-frames;*
- *$\Delta 4$ is valid on the class of all* 4*-frames;*
- *$\Delta 5$ is valid on the class of all* 5*-frames;*
- *ΔB is valid on the class of all $\mathcal{B}$-frames.*

Proof. Take the validity of ΔT and ΔB for example.

Given any reflexive model $\mathcal{M} = \langle S, \{\to_i \mid i \in \mathbf{I}\}, V\rangle$ and any $s \in S$, suppose $\mathcal{M}, s \vDash \Delta_i\varphi \wedge \Delta_i(\varphi \to \psi) \wedge \varphi$. Towards a contradiction assume $\mathcal{M}, s \nvDash \Delta_i\psi$, then there exist t, t' such that $s \to_i t, s \to_i t'$ and $t \vDash \psi, t' \vDash \neg\psi$. From the reflexivity of s it follows that $s \to_i s$, and thus $t \vDash \varphi, t' \vDash \varphi$ by the facts that $s \vDash \Delta_i\varphi \wedge \varphi, s \to_i t, s \to_i t'$. Then $t \vDash \varphi \to \psi$ but $t' \nvDash \varphi \to \psi$, which contradicts the supposition $s \vDash \Delta_i(\varphi \to \psi)$.

Given any symmetric model $\mathcal{M} = \langle S, \{\to_i \mid i \in \mathbf{I}\}, V\rangle$ and any $s \in S$, suppose that $\mathcal{M}, s \vDash \varphi$. Let $t \in S$ and $i \in \mathbf{I}$ with $s \to_i t$. By the symmetry of $\to_i$, we have $t \to_i s$. We show that $\mathcal{M}, t \vDash (\Delta_i\varphi \wedge \Delta_i(\varphi \to \psi) \wedge \neg\Delta_i\psi) \to \chi$. If $\mathcal{M}, t \vDash \Delta_i\varphi \wedge \Delta_i(\varphi \to \psi) \wedge \neg\Delta_i\psi$, then there exist t_1, t_2 such that $t \to_i t_1$ and $t \to_i t_2$ and $t_1 \vDash \psi$ and $t_2 \vDash \neg\psi$. From $t \vDash \Delta_i\varphi$, $t \to_i s$ and the supposition, it follows that $t_1 \vDash \varphi$ and $t_2 \vDash \varphi$. Thus $t_1 \vDash \varphi \to \psi$ and $t_2 \vDash \neg(\varphi \to \psi)$, contrary to the fact that $t \vDash \Delta_i(\varphi \to \psi)$ and $t \to_i t_1$ and $t \to_i t_2$. Therefore $\mathcal{M}, t \nvDash \Delta_i\varphi \wedge \Delta_i(\varphi \to \psi) \wedge \neg\Delta_i\psi$, which implies that $\mathcal{M}, t \vDash (\Delta_i\varphi \wedge \Delta_i(\varphi \to \psi) \wedge \neg\Delta_i\psi) \to \chi$, as desired. □

The following proposition says that $\Delta 4$ and $\Delta 5$ are provable in $\mathbb{CLS}4$ and $\mathbb{CLS}5$ respectively, which are crucial in the proofs of Theorems 5.14 and 5.15, respectively.

PROPOSITION 5.3.

1. $\vdash_{\mathbb{CLS}4} \Delta_i\varphi \to \Delta_i(\Delta_i\varphi \vee \psi)$
2. $\vdash_{\mathbb{CLS}5} \neg\Delta_i\varphi \to \Delta_i(\neg\Delta_i\varphi \vee \psi)$

Proof.

1. The following is a derivation in $\mathbb{CLS}4$:

(i)	$\Delta_i\varphi \to (\Delta_i\varphi \vee \psi)$	TAUT
(ii)	$\Delta_i(\Delta_i\varphi \to (\Delta_i\varphi \vee \psi))$	NECΔ, (i)
(iii)	$\Delta_i\Delta_i\varphi \wedge \Delta_i(\Delta_i\varphi \to (\Delta_i\varphi \vee \psi)) \wedge \Delta_i\varphi \to \Delta_i(\Delta_i\varphi \vee \psi)$	ΔT
(iv)	$\Delta_i\Delta_i\varphi \wedge \Delta_i\varphi \to \Delta_i(\Delta_i\varphi \vee \psi)$	(ii), (iii)
(v)	$\Delta_i\varphi \to \Delta_i\Delta_i\varphi$	wΔ4
(vi)	$\Delta_i\varphi \to \Delta_i(\Delta_i\varphi \vee \psi)$	(iv), (v)

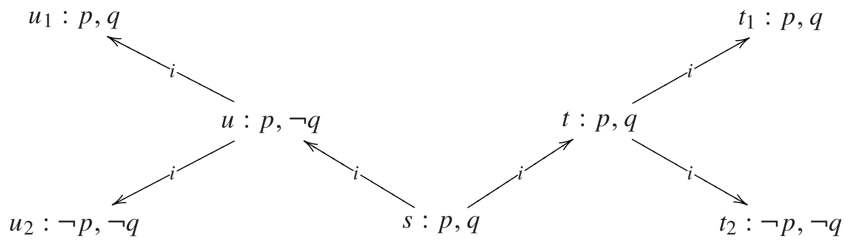
2. Similar to 1, by using Axiom wΔ5. □

Therefore, in the presence of Axiom ΔT, $\Delta 4$ is provable in $\mathbb{CL}$ + wΔ4, and $\Delta 5$ is provable in $\mathbb{CL}$ + wΔ5. One may ask if $\mathbb{CL}$ + wΔ4 (resp. $\mathbb{CL}$ + wΔ5), without ΔT, is enough to axiomatize **CL** over transitive (resp. Euclidean) frames. However, the answers are negative. The proposition below and next proposition were shown, respectively, by George Schumm in a review on Kuhn (1995) in *Mathematical Review* (1996) and by Zolin (1999) using one *frame*. Here we show them with the aid of one *model*.

PROPOSITION 5.4. $\mathbb{CL}$ + *w*Δ4 *is incomplete with respect to the class of transitive frames.*

Proof. Recall that $\Delta_i p \to \Delta_i(\Delta_i p \vee q)$ is an instance of $\Delta 4$ and it is valid on the class of transitive frames (Prop. 5.2). We will show that this formula is not a theorem of $\mathbb{CL}$+wΔ4. For this, we construct a model $\mathcal{M}$ such that $\mathbb{CL}$+wΔ4 is sound with respect to validity on $\mathcal{M}$ (i.e. for any **CL** formula φ, $\vdash_{\mathbb{CL}+\mathrm{w}\Delta 4} \varphi$ implies $\mathcal{M} \vDash \varphi$), but $\mathcal{M} \nvDash \Delta_i p \to \Delta_i(\Delta_i p \vee q)$. Therefore $\nvdash_{\mathbb{CL}+\mathrm{w}\Delta 4} \Delta_i p \to \Delta_i(\Delta_i p \vee q)$. Since $\Delta_i p \to \Delta_i(\Delta_i p \vee q)$ is not provable in $\mathbb{CL}$ + wΔ4 but it is valid over transitive frames, $\mathbb{CL}$ + wΔ4 is not complete w.r.t. the class of transitive frames.

Consider the following model $\mathcal{M}$ (w.l.o.g. let us assume $\mathbf{P} = \{p, q\}$):



First, remember that all the axioms of $\mathbb{CL}$ are valid on the class of all frames (Prop. 4.2), thus they are also valid on $\mathcal{M}$. As for the inference rules, their validities on $\mathcal{M}$ do not follow immediately from the fact that these rules are valid on the class of all frames. However, it is not hard to check that $\mathtt{MP}$, $\mathtt{NEC}\Delta$ and $\mathtt{RE}\Delta$ are indeed valid on $\mathcal{M}$, i.e., if the premise is valid on $\mathcal{M}$ then the conclusion is also valid on $\mathcal{M}$.

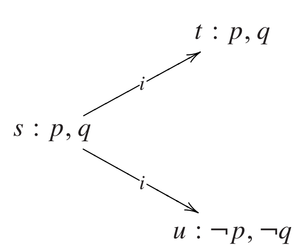
Second, $\mathtt{w}\Delta 4$ is valid on $\mathcal{M}$: by the construction of $\mathcal{M}$, it is not hard to show by induction on the structure of $\varphi \in \mathbf{CL}$ that: for any φ, $t_1 \vDash \varphi$ iff $u_1 \vDash \varphi$, and $t_2 \vDash \varphi$ iff $u_2 \vDash \varphi$ (∗). As none of worlds t_1, t_2, u_1, u_2 has any successor, then all of them satisfy $\Delta_i \Delta_i \varphi$, thus also satisfy $\mathtt{w}\Delta 4$ ($\Delta_i \varphi \to \Delta_i \Delta_i \varphi$). Also, since t_1 and t_2 both satisfy $\Delta_i \varphi$ for any φ, $t \vDash \Delta_i \Delta_i \varphi$ for any φ too, and thus $t \vDash \Delta_i \varphi \to \Delta_i \Delta_i \varphi$. Similarly, we can show that $u \vDash \Delta_i \varphi \to \Delta_i \Delta_i \varphi$ for any φ. Now from (∗) we can see $t \vDash \Delta_i \varphi$ iff $u \vDash \Delta_i \varphi$, which implies $s \vDash \Delta_i \Delta_i \varphi$, and thus $s \vDash \Delta_i \varphi \to \Delta_i \Delta_i \varphi$. In sum, $\mathtt{w}\Delta 4$ is valid on $\mathcal{M}$.

Finally, it is clear that $\mathcal{M}, s \nvDash \Delta_i p \to \Delta_i(\Delta_i p \vee q)$, thus $\mathcal{M} \nvDash \Delta_i p \to \Delta_i(\Delta_i p \vee q)$. Since $\mathbb{CL} + \mathtt{w}\Delta 4$ is sound w.r.t. $\mathcal{M}$ then we have $\nvdash_{\mathbb{CL}+\mathtt{w}\Delta 4} \Delta_i p \to \Delta_i(\Delta_i p \vee q)$. □

PROPOSITION 5.5. *$\mathbb{CL} + \mathtt{w}\Delta 5$ is incomplete with respect to the class of Euclidean frames.*

Proof. The strategy is similar to the one in the proof of Prop. 5.4. Recall that the formula $\neg \Delta_i p \to \Delta_i(\neg \Delta_i p \vee q)$ is valid on the class of all Euclidean frames (Prop. 5.2). We only need to show that this formula is not a theorem of $\mathbb{CL} + \mathtt{w}\Delta 5$. For this, we construct a model $\mathcal{N}$ such that $\mathbb{CL} + \mathtt{w}\Delta 5$ is sound with respect to $\mathcal{N}$ (i.e., all the theorems of $\mathbb{CL} + \mathtt{w}\Delta 5$ are valid on $\mathcal{N}$), but $\mathcal{N} \nvDash \neg \Delta_i p \to \Delta_i(\neg \Delta_i p \vee q)$.

Consider the following model $\mathcal{N}$ (again, let us assume $\mathbf{P} = \{p, q\}$):



As in the previous proof, the axioms and inference rules of $\mathbb{CL}$ are valid on $\mathcal{N}$.

Now we show $\mathtt{w}\Delta 5$ is valid on $\mathcal{N}$: by the construction of $\mathcal{N}$, neither t nor u has successor, then they both satisfy $\Delta_i \neg \Delta_i \varphi$, and thus satisfy $\mathtt{w}\Delta 5$ ($\neg \Delta_i \varphi \to \Delta_i \neg \Delta_i \varphi$). Also, $t \vDash \Delta_i \varphi$ and $u \vDash \Delta_i \varphi$, then $s \vDash \Delta_i \neg \Delta_i \varphi$, and thus $s \vDash \neg \Delta_i \varphi \to \Delta_i \neg \Delta_i \varphi$.

It is clear that $\mathcal{N}, s \nvDash \neg \Delta_i p \to \Delta_i(\neg \Delta_i p \vee q)$, thus $\mathcal{N} \nvDash \neg \Delta_i p \to \Delta_i(\neg \Delta_i p \vee q)$. □

We now continue with the completeness proofs for the extended proof systems. We first address the completeness of $\mathbb{CL}$ over *serial* frames. Note that $\to_i^c$ (for every $i \in \mathbf{I}$) is not necessarily serial, since some states in $\mathcal{M}^c$ may be *endpoints* w.r.t. $\to_i^c$, i.e. the states u which have no $\to_i^c$-successors in S^c (e.g. $\Delta_i\varphi \in u$ for all φ). Hence we need a strategy to transform $\mathcal{M}^c$ into a serial model, while the value of every formula at each point in S^c is preserved. The proof below is simpler than those in Humberstone (1995) and Zolin (1999).

THEOREM 5.6 (Humberstone, 1995; Zolin, 1999). *$\mathbb{CL}$ is complete with respect to the class $\mathcal{D}$ of serial frames.*

Proof. Define $\mathcal{M}^c$ as in Def. 4.5. The strategy is '*reflexivizing the endpoints*': given any $i \in \mathbf{I}$, add one arrow from each endpoint w.r.t. $\to_i^c$ in $\mathcal{M}^c$ to itself. Formally, $\to_i^{\mathbf{D}} = \to_i^c \cup \{(t,t) \mid t$ is an endpoint w.r.t. $\to_i^c$ in $\mathcal{M}^c\}$. Denote the model obtained in this way as $\mathcal{M}^{\mathbf{D}} = \{S^c, \{\to_i^{\mathbf{D}} \mid i \in \mathbf{I}\}, V^c\}$. It is now clear that $\mathcal{M}^{\mathbf{D}}$ is serial from the construction of $\mathcal{M}^{\mathbf{D}}$.

Moreover, for all $\varphi \in \mathbf{CL}$, $\mathcal{M}^c, s \vDash \varphi$ iff $\mathcal{M}^{\mathbf{D}}, s \vDash \varphi$, from which we get the completeness result by Lemma 4.6. The nontrivial case is $\Delta_i\varphi$. If s is a nonendpoint w.r.t. $\to_i^c$, then the claim is clear; if s is an endpoint w.r.t. $\to_i^c$, then $\mathcal{M}^c, s \vDash \Delta_i\varphi$ and $\mathcal{M}^{\mathbf{D}}, s \vDash \Delta_i\varphi$ due to the semantics of $\Delta_i\varphi$. □

We then address the completeness of $\mathbb{CLT}$. In the canonical model construction of Def. 4.5 it is unclear whether the canonical relation is reflexive. To ensure that the relations are reflexive, we take the reflexive closure of the canonical relation.

DEFINITION 5.7 (Canonical model of $\mathbb{CLT}$). *The canonical model $\mathcal{M}^{\boldsymbol{T}} = \langle S^c, \{\to_i^{\boldsymbol{T}} \mid i \in \boldsymbol{I}\}, V^c\rangle$ of $\mathbb{CLT}$ is the same as $\mathcal{M}^c$ in Def. 4.5, except that S^c consists of all maximal consistent sets* of $\mathbb{CLT}$, *and that $\to_i^{\boldsymbol{T}}$ is* the reflexive closure of $\to_i^c$ *defined in Def. 4.5.*

LEMMA 5.8 (Truth Lemma for $\mathbb{CLT}$). *For any **CL** formula φ, $\mathcal{M}^{\boldsymbol{T}}, s \vDash \varphi$ iff $\varphi \in s$.*

Proof. By induction on φ. We consider the nontrivial case for $\Delta_i\varphi$.

Left-to-right: This is similar to the proof for 'Only if' in Lemma 4.6. Observe that all pairs in the relation $\to_i^c$ in Def. 4.5 are also in the relation $\to_i^{\mathbf{T}}$ from Def. 5.7.

Right-to-left: Assume towards contradiction that $\Delta_i\varphi \in s$ but $\mathcal{M}^{\mathbf{T}}, s \vDash \neg\Delta_i\varphi$, namely, there exist $t, u \in S^c$ such that $s \to_i^{\mathbf{T}} t$ and $s \to_i^{\mathbf{T}} u$ and $\mathcal{M}^{\mathbf{T}}, t \vDash \varphi$ and $\mathcal{M}^{\mathbf{T}}, u \vDash \neg\varphi$. By induction hypothesis, $\varphi \in t$ and $\neg\varphi \in u$. As $\to_i^{\mathbf{T}}$ is reflexive, we only need to consider two cases (the case $s = t$ and $s = u$ is impossible, because $t \neq u$):

- $s \neq t$ and $s \neq u$. Then $s \to_i^c t$ and $s \to_i^c u$, and thus the proof is same as the proof for 'If' in Lemma 4.6. And finally we can get a contradiction.
- Either $s = t$ or $s = u$. Without loss of generality, we may as well consider the case $s = t$ (thus $\varphi \in s$) and $s \neq u$. From $s \to_i^{\mathbf{T}} u$ and $s \neq u$ it follows that $s \to_i^c u$, thus there exists χ such that $\neg\Delta_i\chi \in s$ and (†): for all φ, $\Delta_i\varphi \wedge \Delta_i(\chi \to \varphi) \in s$ implies $\varphi \in u$. Now since $\neg\varphi \in u$ and $\Delta_i\varphi \in s$, by (†) we have $\neg\Delta_i(\chi \to \varphi) \in s$, i.e., $\neg\Delta_i(\neg\varphi \to \neg\chi) \in s$. Using Axiom $\Delta\mathtt{T}$ and the fact that $\Delta_i\varphi \wedge \varphi \in s$, we get $\Delta_i(\varphi \to \chi) \to \Delta_i\chi \in s$, thus $\neg\Delta_i(\varphi \to \chi) \in s$. Then $\neg\Delta_i\varphi \in s$ follows from Axiom $\Delta\mathtt{Dis}$ and $\neg\Delta_i(\neg\varphi \to \neg\chi) \in s$, which contradicts the assumption $\Delta_i\varphi \in s$ and the consistency of s, as desired. □

Based on the above lemma, it is routine to show the following.

THEOREM 5.9. $\mathbb{CLT}$ *is complete with respect to the class of all* $\mathcal{T}$*-frames.*

Now let us look at the completeness for $\mathbb{CL}4$ and $\mathbb{CL}5$. In these cases we do not need to revise the canonical relations. The proof is different from those in Kuhn (1995) and Zolin (1999).

THEOREM 5.10 (Kuhn, 1995; Zolin, 1999). $\mathbb{CL}4$ *is complete with respect to the class of all* 4*-frames.*

Proof. Define $\mathcal{M}^c$ as in Def. 4.5 w.r.t. $\mathbb{CL}4$. We only need to show that $\to_i^c$ is transitive.

Given $s, t, u \in S^c$. Assume that $s \to_i^c t$ and $t \to_i^c u$, the only thing is to show that $s \to_i^c u$. From $s \to_i^c t$ it follows that for some χ such that $\neg\Delta_i\chi \in s$ and $(*)$: for all φ, $\Delta_i\varphi \wedge \Delta_i(\chi \to \varphi) \in s$ implies $\varphi \in t$. From $t \to_i^c u$ it follows that for some ψ such that $\neg\Delta_i\psi \in t$ and $(\star)$: for all φ, $\Delta_i\varphi \wedge \Delta_i(\psi \to \varphi) \in t$ implies $\varphi \in u$. To show $s \to_i^c u$, by the definition of $\to_i^c$, we need to prove that for all φ, $\Delta_i\varphi \wedge \Delta_i(\chi \to \varphi) \in s$ implies $\varphi \in u$. Now fixing a formula φ such that $\Delta_i\varphi \wedge \Delta_i(\chi \to \varphi) \in s$, we need to show $\varphi \in u$. If we can show that $\Delta_i\varphi \wedge \Delta_i(\psi \to \varphi) \in t$, then by $(\star)$, we have $\varphi \in u$.

We first show that $\Delta_i\varphi \in t$: since $\Delta_i\varphi \in s$, first, by $\mathtt{w}\Delta 4$, we have $\Delta_i\Delta_i\varphi \in s$; second, by Axiom $\Delta\mathtt{4}$, we get $\Delta_i(\Delta_i\varphi \vee \neg\chi) \in s$, i.e., $\Delta_i(\chi \to \Delta_i\varphi) \in s$. We have thus proved that $\Delta_i\Delta_i\varphi \wedge \Delta_i(\chi \to \Delta_i\varphi) \in s$, then by $(*)$, we have $\Delta_i\varphi \in t$.

We now show that $\Delta_i(\psi \to \varphi) \in t$: as $\neg\Delta_i\chi \in s$, it follows from Axiom $\Delta\leftrightarrow$ that $\neg\Delta_i\neg\chi \in s$. Since $\Delta_i(\chi \to \varphi) \in s$, we have $\Delta_i(\neg\varphi \to \neg\chi) \in s$. Then by Axiom $\Delta\mathtt{Con}$, we obtain $\neg\Delta_i(\varphi \to \neg\chi) \in s$. Since $\Delta_i\varphi \in s$, $\Delta_i(\varphi \to \neg\chi) \vee \Delta_i(\neg\varphi \to \neg\psi) \in s$ by Axiom $\Delta\mathtt{Dis}$, thus $\Delta_i(\neg\varphi \to \neg\psi) \in s$, i.e., $\Delta_i(\psi \to \varphi) \in s$. Using $\mathtt{w}\Delta 4$ again, we obtain $\Delta_i\Delta_i(\psi \to \varphi) \in s$; using Axiom $\Delta\mathtt{4}$ again, we get $\Delta_i(\Delta_i(\psi \to \varphi) \vee \neg\chi) \in s$, i.e., $\Delta_i(\chi \to \Delta_i(\psi \to \varphi)) \in s$. Now $\Delta_i\Delta_i(\psi \to \varphi) \wedge \Delta_i(\chi \to \Delta_i(\psi \to \varphi)) \in s$. From $(*)$, we conclude that $\Delta_i(\psi \to \varphi) \in t$. □

THEOREM 5.11 (Zolin, 1999). $\mathbb{CL}5$ *is complete with respect to the class of all* 5*-frames.*

Proof. Define $\mathcal{M}^c$ as in Def. 4.5 w.r.t. $\mathbb{CL}5$. We only need to show that $\to_i^c$ is Euclidean.

Given $s, t, u \in S^c$. Suppose that $s \to_i^c t$ and $s \to_i^c u$, the only thing is to show that $t \to_i^c u$. From $s \to_i^c t$ it follows that for some χ such that $\neg\Delta_i\chi \in s$ and $(*)$: for all φ, $\Delta_i\varphi \wedge \Delta_i(\chi \to \varphi) \in s$ implies $\varphi \in t$. From $s \to_i^c u$ it follows that for some ψ such that $\neg\Delta_i\psi \in s$ and $(\star)$: for all φ, $\Delta_i\varphi \wedge \Delta_i(\psi \to \varphi) \in s$ implies $\varphi \in u$. To show $t \to_i^c u$, by definition of $\to_i^c$, we need to prove that there exists θ such that:

1. $\neg\Delta_i\theta \in t$, and
2. for all φ, $\Delta_i\varphi \wedge \Delta_i(\theta \to \varphi) \in t$ implies $\varphi \in u$.

We show χ is the desired θ.

For item 1: since $\neg\Delta_i\chi \in s$, first, by $\mathtt{w}\Delta 5$, we have $\Delta_i\neg\Delta_i\chi \in s$; second, by Axiom $\Delta\mathtt{5}$, we get $\Delta_i(\neg\Delta_i\chi \vee \neg\chi) \in s$, i.e., $\Delta_i(\chi \to \neg\Delta_i\chi) \in s$. We have thus shown that $\Delta_i\neg\Delta_i\chi \wedge \Delta_i(\chi \to \neg\Delta_i\chi) \in s$, then by $(*)$, we have $\neg\Delta_i\chi \in t$.

For item 2: fixing a φ, we assume that $\Delta_i\varphi \wedge \Delta_i(\chi \to \varphi) \in t$. We only need to show $\varphi \in u$. Since $\Delta_i\varphi \in t$, i.e., $\neg\Delta_i\varphi \notin t$, by $(*)$ we infer that $\Delta_i\neg\Delta_i\varphi \wedge \Delta_i(\chi \to \neg\Delta_i\varphi) \notin s$. Using $\mathtt{w}\Delta 5$ and Axiom $\Delta\mathtt{5}$ we deduce that $\neg\Delta_i\varphi \notin s$, i.e., $\Delta_i\varphi \in s$. Similarly, from $\Delta_i(\chi \to \varphi) \in t$ we get $\Delta_i(\chi \to \varphi) \in s$, i.e., $\Delta_i(\neg\varphi \to \neg\chi) \in s$. Then by Axiom $\Delta\mathtt{Con}$ and $\neg\Delta_i\neg\chi \in s$ (since $\neg\Delta_i\chi \in s$), we have $\neg\Delta_i(\varphi \to \neg\chi) \in s$. From Axiom $\Delta\mathtt{Dis}$ and the fact that $\Delta_i\varphi \in s$, it follows that $\Delta_i(\neg\varphi \to \neg\psi) \in s$, i.e., $\Delta_i(\psi \to \varphi) \in s$. We have thus proved that $\Delta_i\varphi \wedge \Delta_i(\psi \to \varphi) \in s$. Therefore $\varphi \in u$ follows from ($\star$). □

THEOREM 5.12. $\mathbb{CL}45$ *is complete with respect to the class of all* 45*-frames.*

Proof. This follows directly from Theorems 5.10 and 5.11. The canonical model w.r.t. $\mathbb{CL}45$ is both transitive and Euclidean. □

Actually, the same story applies to $\mathcal{KD}45$-frames, with slight complication. Before we go into the details, let us show the difficulty to arise. Although $\to_i^c$ (for every $i \in \mathbf{I}$) is transitive and Euclidean as shown, it cannot be guaranteed to be serial (see the remark preceding Thm. 5.6). Hence we need a strategy to transform $\mathcal{M}^c$ into a serial model, retaining the properties of transitivity and Euclidicity, while the value of every formula at each point in S^c is preserved.

THEOREM 5.13. $\mathbb{CL}45$ *is complete with respect to the class of all* $\mathcal{KD}45$*-frames.*

Proof. Define $\mathcal{M}^{\mathbf{D}}$ as in Thm. 5.6. We only need to show $\to_i^{\mathbf{D}}$ is transitive and Euclidean.

Transitivity: given any $s, t, u \in S^c$, suppose that $s \to_i^{\mathbf{D}} t$ and $t \to_i^{\mathbf{D}} u$. If $s \neq t$ and $t \neq u$, then $s \to_i^c t$ and $t \to_i^c u$, and the proof thus continues similarly to Thm. 5.10, we can get $s \to_i^c u$, then $s \to_i^{\mathbf{D}} u$. If $s = t$ or $t = u$, then clearly $s \to_i^{\mathbf{D}} u$.

Euclidicity: Given any $s, t, u \in S^c$, suppose that $s \to_i^{\mathbf{D}} t$ and $s \to_i^{\mathbf{D}} u$. If $s \neq t$ and $s \neq u$, then $s \to_i^c t$ and $s \to_i^c u$, and the proof thus continues similarly to Thm. 5.11, we can get $t \to_i^c u$, then $t \to_i^{\mathbf{D}} u$. If $s = t$, clearly $t \to_i^{\mathbf{D}} u$. The only case is $s \neq t$ (thus $s \to_i^c t$) and $s = u$. This case implies $s \to_i^c u$, otherwise s is an endpoint w.r.t. $\to_i^c$, contrary to $s \to_i^c t$. Then the proof is reduced to the first case, and we can get $t \to_i^{\mathbf{D}} u$. □

THEOREM 5.14. $\mathbb{CLS}4$ *is complete with respect to the class of all* $\mathcal{S}4$*-frames.*

Proof. Define $\mathcal{M}^{\mathbf{T}}$ as Def. 5.7 w.r.t. $\mathbb{CLS}4$. Given Thm. 5.9, we only need to show that $\to_i^{\mathbf{T}}$ is transitive. Now given $s, t, u \in S^c$, and assume $s \to_i^{\mathbf{T}} t$ and $t \to_i^{\mathbf{T}} u$, we need to show $s \to_i^{\mathbf{T}} u$. If $s = t$ or $t = u$, then by assumption, we get $s \to_i^{\mathbf{T}} u$. Thus we consider the case $s \neq t$ and $t \neq u$. Then $s \to_i^c t$ and $t \to_i^c u$. The proof for this case is the same as Thm. 5.10, as we can use $\Delta 4$ due to Prop. 5.3. □

THEOREM 5.15. $\mathbb{CLS}5$ *is complete with respect to the class of all* $\mathcal{S}5$*-frames.*

Proof. Define $\mathcal{M}^{\mathbf{T}}$ as Def. 5.7 w.r.t. $\mathbb{CLS}5$. Given Thm. 5.9, we only need to show that $\to_i^{\mathbf{T}}$ is Euclidean. Now given $s, t, u \in S^c$, and assume $s \to_i^{\mathbf{T}} t$ and $s \to_i^{\mathbf{T}} u$, we need to show $t \to_i^{\mathbf{T}} u$. If $s \neq t$ and $s \neq u$, then $s \to_i^c t$ and $s \to_i^c u$, and the proof is the same as in Thm. 5.11, as we can use $\Delta 5$ due to Prop. 5.3. If $s = t$, then by the assumption, we get $t \to_i^{\mathbf{T}} u$. The only case to consider is $s = u$ and $s \neq t$ (thus $s \to_i^c t$), to show $t \to_i^{\mathbf{T}} u$. Analogous to the corresponding proof of Thm. 5.11, we can show that $\neg\Delta_i\chi \in t$ (item 1); for all φ, suppose $\Delta_i\varphi \wedge \Delta_i(\chi \to \varphi) \in t$, we can derive $\Delta_i\neg\varphi \wedge \Delta_i(\neg\varphi \to \neg\chi) \wedge \neg\Delta_i\neg\chi \in s$, then using Axiom $\Delta\mathrm{T}$, we get $\neg\varphi \notin s$, i.e., $\varphi \in s$, that is, $\varphi \in u$ (item 2). Thus $t \to_i^c u$, therefore $t \to_i^{\mathbf{T}} u$. □

§6. Axiomatization over symmetric frames. The completeness proof of system $\mathbb{CLB}$ (see Def. 5.1) over symmetric frames is quite involved, which is worth presenting in a single section. As claimed in the footnote 6, unlike $\mathbb{CL}$, $\mathrm{NEC}\Delta$ is admissible in $\mathbb{CLB} - \mathrm{NEC}\Delta$. This means that $\mathbb{CLB}$ can be replaced with $\mathbb{CLB} - \mathrm{NEC}\Delta$.

PROPOSITION 6.1. *$NEC\Delta$ is admissible in $\mathbb{CLB} - NEC\Delta$.*[7]

Proof. Suppose that $\vdash \varphi$. Then by axiom ΔB, we have $\vdash \Delta_i((\Delta_i\varphi \wedge \Delta_i(\varphi \to \psi) \wedge \neg\Delta_i\psi) \to \top)$. Then by TAUT and RE$\Delta$, we obtain $\vdash \Delta_i\top$. Using supposition and TAUT again, we deduce $\vdash \varphi \leftrightarrow \top$. Then applying RE$\Delta$ again, we get $\vdash \Delta_i\varphi \leftrightarrow \Delta_i\top$, thus $\vdash \Delta_i\varphi$. □

The following proposition states that $\mathcal{M}^c$ in Def. 4.5 is *almost* symmetric. Note that it is equivalent to the multi-agent version of Fan *et al.* (2014, Prop. 5.8), by using the definition of $\to_i^c$ and the 'only if' part of Lemma 4.6.

PROPOSITION 6.2. *For any $s, t \in S^c$ and any $i \in \mathbf{I}$, if $s \to_i^c t$ and $t \to_i^c t'$ for some $t' \in S^c$, then $t \to_i^c s$.*

Proof. Assume that $s \to_i^c t$ and $t \to_i^c t'$ for some $t' \in S^c$ (thus $\neg\Delta_i\chi \in t$ for some χ), we need to show $t \to_i^c s$. Suppose not, then there exists φ such that $\Delta_i\varphi \wedge \Delta_i(\chi \to \varphi) \in t$ but $\varphi \notin s$ (thus $\neg\varphi \in s$). Since $s \to_i^c t$, by definition, there is a ψ such that $\neg\Delta_i\psi \in s$ and ($\star$): for all $\theta : \Delta_i\theta \wedge \Delta_i(\psi \to \theta) \in s$ implies $\theta \in t$. Thanks to ΔB, since $\neg\varphi \in s$, we have $\Delta_i((\Delta_i\neg\varphi \wedge \Delta_i(\neg\varphi \to \neg\chi) \wedge \neg\Delta_i\neg\chi) \to \neg\psi) \in s$ and $\Delta_i(\Delta_i\neg\varphi \wedge \Delta_i(\neg\varphi \to \neg\chi) \wedge \neg\Delta_i\neg\chi) \in s$.[8] By $\Delta\leftrightarrow$ and REΔ, $\Delta_i(\Delta_i\varphi \wedge \Delta_i(\chi \to \varphi) \wedge \neg\Delta_i\chi) \wedge \Delta_i((\Delta_i\varphi \wedge \Delta_i(\chi \to \varphi) \wedge \neg\Delta_i\chi) \to \neg\psi) \in s$, finally we have $\Delta_i\neg(\Delta_i\varphi \wedge \Delta_i(\chi \to \varphi) \wedge \neg\Delta_i\chi) \wedge \Delta_i(\psi \to \neg(\Delta_i\varphi \wedge \Delta_i(\chi \to \varphi) \wedge \neg\Delta_i\chi)) \in s$. By ($\star$), we have $\neg(\Delta_i\varphi \wedge \Delta_i(\chi \to \varphi) \wedge \neg\Delta_i\chi) \in t$, contradiction. □

Note that the canonical model in the unimodal case for $\mathbb{CLB}$ (Fan *et al.*, 2014, Def. 5.9) cannot be generalized into the multimodal case, since the dead ends therein are now relative to the agents. For example, a dead end for agent j may be not a dead end for agent i. Thus we need a new strategy to turn $\mathcal{M}^c$ into a symmetric model, while unchanging the truth values of formulas.

The strategy is as follows. We enumerate all of the agents in $\mathbf{I}$ as $1, 2, 3, \cdots, m$. Starting from $\mathcal{M}^0 = \mathcal{M}^c$ (we may as well assume that $\mathcal{M}^c$ has run out of Prop. 6.2), we construct the desired model (call it $\mathcal{M}^m$) in m steps. In each step we tackle the *dead ends* for that agent, i.e. the states which have incoming but no outgoing transitions for that agent in the previous step, by replacing those dead ends with some new copies of themselves such that each copy has only one incoming transition for that agent and then adding the back arrows for the agent, while keeping all the arrows for the other agents in place, with corresponding replacements for the dead ends. We have to provide that in each step, the accessibility relation for that agent is symmetric (Lemma 6.9), and the symmetry of the previous relation for a fixed agent is not broken (Lemma 6.10), which guarantee $\mathcal{M}^m$ to be symmetric (Prop. 6.8). Moreover, each step preserves the truth values of formulas (Prop. 6.11).

Before giving the formal definition of the canonical model of $\mathbb{CLB}$, we first introduce some useful notation. Let $\mathcal{M}^n = \langle S^n, \{\to_i^n \mid i \in \mathbf{I}\}, V^n\rangle$. By $\mathcal{M}^n$ we mean the obtained model from $\mathcal{M}^c$ after the construction of n-th step. By $\to_i^n$ (where n and i may be equal) we mean the accessibility relation for *agent i* at n-th *step*. Usually, n ranges over $[0, m]$ and i ranges over $[1, m]$ unless mentioned particularly. By D_n we mean the dead ends for

[7] In the proof of this proposition, by abuse of notation, we use $\vdash \varphi$ to denote that φ is provable in $\mathbb{CLB} - \text{NEC}\Delta$.

[8] Note that if we let χ be $\bot$ in ΔB then we obtain a restriction of ΔB which will give us the latter under REΔ and $\Delta\leftrightarrow$.

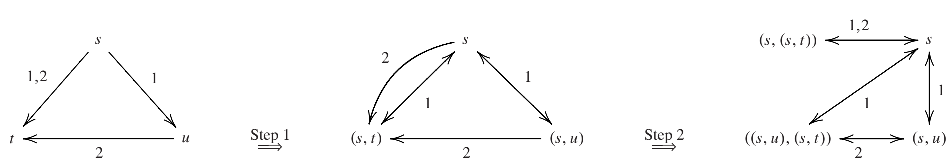
agent n in $\mathcal{M}^{n-1}$, formally $D_n = \{t \mid t \in S^{n-1}, s \to_n^{n-1} t$ for some $s \in S^{n-1}$ and $t \to_n^{n-1} t'$ for no $t' \in S^{n-1}\}$. Let $\bar{D}_n = S^{n-1} \backslash D_n$. For convenience, we rephrase the semantics of $\Delta_i \varphi$ in $(\mathcal{M}^n, s)$ as the following:

$$\mathcal{M}^n, s \vDash \Delta_i \varphi \iff \text{for all } t_1, t_2 \in S^n, \text{ if } s \to_i^n t_1 \text{ and } s \to_i^n t_2, \text{ then } t_1 \vDash \varphi \text{ iff } t_2 \vDash \varphi.$$

DEFINITION 6.3 (Canonical model of $\mathbb{CLB}$). *The* canonical model $\mathcal{M}^m$ *of* $\mathbb{CLB}$ *is a tuple* $\langle S^m, \{\to_i^m \mid i \in \boldsymbol{I}\}, f^m, V^m \rangle$ *defined by induction on* $n \leq m$*:*

- $S^0 = S^c$, $\to_i^0 = \to_i^c$.
- $S^n = \bar{D}_n \cup \{(s,t) \mid t \in D_n$ *and* $s \to_n^{n-1} t\}$
- $s \to_n^n t$ *iff one of the following cases holds:*
    1. $s, t \in \bar{D}_n$ *and* $s \to_n^{n-1} t$,
    2. $s \in \bar{D}_n$ *and* $t = (s, s') \in S^n$,
    3. $t \in \bar{D}_n$ *and* $s = (t, t') \in S^n$.
- *For* $i \neq n$, $s \to_i^n t$ *iff one of the following cases holds:*
    1. $s, t \in \bar{D}_n$ *and* $s \to_i^{n-1} t$,
    2. $s \in \bar{D}_n$ *and* $t = (s'', s') \in S^n$ *and* $s \to_i^{n-1} s'$,
    3. $t \in \bar{D}_n$ *and* $s = (t'', t') \in S^n$ *and* $t' \to_i^{n-1} t$,
    4. $s = (w, v) \in S^n$ *and* $t = (w', v') \in S^n$ *and* $v \to_i^{n-1} v'$.
- f^{n+1} *is a function from* S^{n+1} *to* S^n *such that* $f^{n+1}(s) = s$ *for* $s \in \bar{D}_{n+1}$, *and* $f^{n+1}((s,t)) = t$ *for* $(s,t) \in S^{n+1}$.
- $V^0(p) = \{s \in S^c \mid p \in s\}$ *and* $V^{n+1}(p) = \{s \in S^{n+1} \mid f^{n+1}(s) \in V^n(p)\}$

It is instructive to give a concrete example, as below.



The above sequence of models $\mathcal{M}^0 (= \mathcal{M}^c), \mathcal{M}^1, \mathcal{M}^2$ (in order) indicates how to turn a nonsymmetric model into a symmetric model, in two steps. In Step 1, we tackle the dead ends for agent 1. We can see that the dead ends for that agent in $\mathcal{M}^0$ consist of t and u, moreover, $s \to_1^0 t$ and $s \to_1^0 u$, hence the states t and u (in $\mathcal{M}^0$) are replaced by the copies (s,t) and (s,u) (in $\mathcal{M}^1$) respectively, such that each copy has only one incoming transition for agent 1, and then add the back arrows for that agent. At the same time, all the arrows for the other agents (agent 2, for that matter) are kept in place, with corresponding replacements for the dead ends. The similar analysis goes with Step 2.

First, Propositions 6.4–6.6 indicate the properties of functions f^{n+1}.

PROPOSITION 6.4. *For every* $n \in [0, m-1]$, f^{n+1} *is surjective.*

Proof. Given any $t \in S^n$, we need to show that there exists $u \in S^{n+1}$ such that $f^{n+1}(u) = t$.

If $t \in \bar{D}_{n+1}$, then by definition, we have $f^{n+1}(t) = t$, and $t \in S^{n+1}$; if $t \in D_{n+1}$, by definition of D_{n+1}, there exists $s \in S^n$ such that $s \to_{n+1}^n t$, thus $(s,t) \in S^{n+1}$, then the definition of f^{n+1} implies $f^{n+1}((s,t)) = t$. Either case implies that there exists $u \in S^{n+1}$ such that $f^{n+1}(u) = t$. □

PROPOSITION 6.5 (Preservation). *Given any $s, t \in S^{n+1}$. If $f^{n+1}(s) \to_i^n f^{n+1}(t)$, then*

1. *If $i \neq n+1$, then $s \to_i^{n+1} t$.*
2. *If $i = n+1$, then for some $t' \in S^{n+1}$ such that $s \to_i^{n+1} t'$ and $f^{n+1}(t) = f^{n+1}(t')$.*

Proof. Suppose that $f^{n+1}(s) \to_i^n f^{n+1}(t)$.
For 1: assume that $i \neq n+1$. Consider four cases:

- $s, t \in \bar{D}_{n+1}$. Then $f^{n+1}(s) = s$ and $f^{n+1}(t) = t$. Thus $s \to_i^n t$, hence $s \to_i^{n+1} t$.
- $s \in \bar{D}_{n+1}$ and $t \notin \bar{D}_{n+1}$. Then $t = (u', u) \in S^{n+1}$ (thus $f^{n+1}(t) = u$) and $f^{n+1}(s) = s$. From supposition it follows that $s \to_i^n u$, thus $s \to_i^{n+1} t$.
- $t \in \bar{D}_{n+1}$ and $s \notin \bar{D}_{n+1}$. Then $f^{n+1}(t) = t$ and $s = (u', u) \in S^{n+1}$, thus $f^{n+1}(s) = u$. From supposition it follows that $u \to_i^n t$, thus $s \to_i^{n+1} t$.
- $s, t \notin \bar{D}_{n+1}$. Then $s = (w, v) \in S^{n+1}$ and $t = (w', v') \in S^{n+1}$, thus $f^{n+1}(s) = v$ and $f^{n+1}(t) = v'$. By supposition, $v \to_i^n v'$, thus $s \to_i^{n+1} t$.

For 2: assume that $i = n+1$. Similarly, consider four cases: case 1 is similar to the previous item 1, cases 3 and 4 will lead to contradictions to the supposition. The nontrivial case is case 2, i.e., $s \in \bar{D}_{n+1}$ and $t \notin \bar{D}_{n+1}$. Let $f^{n+1}(t) = u$ and we have $s \to_i^n u$ since $f^{n+1}(s) = s$. Then $(s, u) \in S^{n+1}$, thus $s \to_i^{n+1} (s, u)$, moreover, $f^{n+1}(t) = u = f^{n+1}((s, u))$. Note that t may not be the same as (s, u). □

PROPOSITION 6.6 (No Miracle). *Given any $s, t \in S^{n+1}$.*

1. *If $i \neq n+1$, then $s \to_i^{n+1} t$ implies $f^{n+1}(s) \to_i^n f^{n+1}(t)$.*
2. *If $i = n+1$ and $s \in \bar{D}_{n+1}$, then $s \to_i^{n+1} t$ implies $f^{n+1}(s) \to_i^n f^{n+1}(t)$.*

Proof. For 1: assume that $i \neq n+1$ and $s \to_i^{n+1} t$. By definition, we consider four cases:

- $s, t \in \bar{D}_{n+1}$ and $s \to_i^n t$. Then $f^{n+1}(s) = s$ and $f^{n+1}(t) = t$, thus $f^{n+1}(s) \to_i^n f^{n+1}(t)$.
- $s \in \bar{D}_{n+1}$ and $t = (s'', s') \in S^{n+1}$ and $s \to_i^n s'$. Then $f^{n+1}(s) = s$ and $f^{n+1}(t) = s'$, thus $f^{n+1}(s) \to_i^n f^{n+1}(t)$.
- $t \in \bar{D}_{n+1}$ and $s = (t'', t') \in S^{n+1}$ and $t' \to_i^n t$. Then $f^{n+1}(s) = t'$ and $f^{n+1}(t) = t$, thus $f^{n+1}(s) \to_i^n f^{n+1}(t)$.
- $s = (w, v) \in S^{n+1}$ and $t = (w', v') \in S^{n+1}$ and $v \to_i^n v'$. Then $f^{n+1}(s) = v$ and $f^{n+1}(t) = v'$, thus $f^{n+1}(s) \to_i^n f^{n+1}(t)$.

For 2: suppose that $i = n+1$, $s \in \bar{D}_{n+1}$ and $s \to_i^{n+1} t$. If $t \in \bar{D}_{n+1}$ and $s \to_i^n t$, then clearly $f^{n+1}(s) \to_i^n f^{n+1}(t)$; otherwise if $t = (s, s') \in S^{n+1}$, then $s \to_{n+1}^n s'$, i.e., $f^{n+1}(s) \to_i^n f^{n+1}(t)$. □

The proposition below states that $\to_i^n$ is *almost* symmetric, which is a generalization of Prop. 6.2.

PROPOSITION 6.7. *For all $n \in [0, m]$, if $s, t \in S^n$, $s \to_i^n t$ and $t \to_i^n t'$ for some $t' \in S^n$, then $t \to_i^n s$.*

Proof. By induction on n. The base case holds by Prop. 6.2. Suppose the statement holds for the case $n = k$. Consider the case $n = k + 1$. Assume that $s, t \in S^{k+1}$, $s \to_i^{k+1} t$ and $t \to_i^{k+1} t'$ for some $t' \in S^{k+1}$, to show $t \to_i^{k+1} s$.

If $i = k + 1$, then by definition, we need to consider the following cases:

- $s, t \in \bar{D}_{k+1}$. Then by Prop. 6.6.2, $f^{k+1}(s) \to_i^k f^{k+1}(t) \to_i^k f^{k+1}(t')$, i.e., $s \to_i^k t \to_i^k f^{k+1}(t')$. By IH, $t \to_i^k s$. Thus $t \to_i^{k+1} s$.
- $s \in \bar{D}_{k+1}$ and $t = (s, s') \in S^{k+1}$. From item 3 of the definition of $\to_{k+1}^{k+1}$, we have $t \to_i^{k+1} s$.
- $t \in \bar{D}_{k+1}$ and $s = (t, t'') \in S^{k+1}$. From item 2 of the definition of $\to_{k+1}^{k+1}$, we have $t \to_i^{k+1} s$.

If $i \neq k + 1$, by Prop. 6.6.1, we have $f^{k+1}(s) \to_i^k f^{k+1}(t) \to_i^k f^{k+1}(t')$. By IH, $f^{k+1}(t) \to_i^k f^{k+1}(s)$. Then by Prop. 6.5.1, $t \to_i^{k+1} s$. □

The model $\mathcal{M}^m$ is indeed symmetric: at the m-th step, all relations $\to_i^m$ are symmetric.

PROPOSITION 6.8. *$\mathcal{M}^m$ is symmetric. That is, for all $i \in [1, m]$, $\to_i^m$ is symmetric.*

To show the above proposition, we show that (a) every relation $\to_n$ at n-th step is symmetric, and (b) the property of symmetry is preserved at every step of construction.

LEMMA 6.9. *For every $n \in [1, m]$, $\to_n^n$ is symmetric.*

Proof. Given any $s, t \in S^n$, suppose that $s \to_n^n t$, we need to show that $t \to_n^n s$. From the supposition and the definition of $\to_n^n$, we consider three cases:

- $s, t \in \bar{D}_n$ and $s \to_n^{n-1} t$ (thus $s, t \in S^{n-1}$). Since $t \in \bar{D}_n$, it follows that $t \notin D_n$. By definition of D_n, we have $t \to_n^{n-1} t'$ for some $t' \in S^{n-1}$, thus $t \to_n^{n-1} s$ by Prop. 6.7. Therefore $t \to_n^n s$.
- $s \in \bar{D}_n$ and $t = (s, s') \in S^n$. By item 3 of the definition of $\to_n^n$, we have $t \to_n^n s$.
- $t \in \bar{D}_n$ and $s = (t, t') \in S^n$. By item 2 of the definition of $\to_n^n$, we have $t \to_n^n s$.

□

LEMMA 6.10. *If $\to_i^n$ is symmetric, then $\to_i^{n+1}$ is also symmetric.*

Proof. Suppose that $\to_i^n$ is symmetric. Assume that $s, t \in S^{n+1}$ with $s \to_i^{n+1} t$. We need to show that $t \to_i^{n+1} s$. The case for $i = n + 1$ is shown in Lemma 6.9. If $i \neq n + 1$, then by Prop. 6.6.1, $f^{n+1}(s) \to_i^n f^{n+1}(t)$. By supposition, we have $f^{n+1}(t) \to_i^n f^{n+1}(s)$. By Prop. 6.5.1, we conclude that $t \to_i^{n+1} s$. □

Thus we complete the proof of Prop. 6.8. We next show that at every step of construction, the truth values of formulas are unchanged.

PROPOSITION 6.11. *For any $n \in [0, m - 1]$, any $s \in S^{n+1}$, and any $\varphi \in \boldsymbol{CL}$,*

$$\mathcal{M}^{n+1}, s \vDash \varphi \iff \mathcal{M}^n, f^{n+1}(s) \vDash \varphi.$$

Proof. Given $s \in S^{n+1}$. By induction on φ, we only consider the nontrivial cases $\varphi = p \in \mathbf{P}$ and $\varphi = \Delta_i \psi$.

- $\varphi = p \in \mathbf{P}$. By definition of V^n, $\mathcal{M}^{n+1}, s \vDash p$ iff $s \in V^{n+1}(p)$ iff $f^{n+1}(s) \in V^n(p)$ iff $\mathcal{M}^n, f^{n+1}(s) \vDash p$.
- $\varphi = \Delta_i \psi$. We show that $\mathcal{M}^{n+1}, s \nvDash \Delta_i \psi \Longleftrightarrow \mathcal{M}^n, f^{n+1}(s) \nvDash \Delta_i \psi$.
  $\Rightarrow$: Suppose that $\mathcal{M}^{n+1}, s \nvDash \Delta_i \psi$, then there exist t_1, t_2 such that $s \to_i^{n+1} t_1$ and $s \to_i^{n+1} t_2$ and $t_1 \vDash \psi$ and $t_2 \nvDash \psi$. If $i = n+1$ and $s \notin \bar{D}_{n+1}$, then by definition of $\to_{n+1}^{n+1}$, we have $s = (t_1, t_1') = (t_2, t_2')$, then $t_1 = t_2$, contradiction. Thus $i \neq n+1$ or ($i = n+1$ and $s \in \bar{D}_{n+1}$). In these two cases, by Prop. 6.6, we have $f^{n+1}(s) \to_i^n f^{n+1}(t_1)$ and $f^{n+1}(s) \to_i^n f^{n+1}(t_2)$; by $t_1 \vDash \psi$ and $t_2 \nvDash \psi$ and IH, we obtain that $f^{n+1}(t_1) \vDash \psi$ and $f^{n+1}(t_2) \nvDash \psi$. Therefore $\mathcal{M}^n, f^{n+1}(s) \nvDash \Delta_i \psi$.
  $\Leftarrow$: Suppose that $\mathcal{M}^n, f^{n+1}(s) \nvDash \Delta_i \psi$, then there exist u_1 and u_2 such that $f^{n+1}(s) \to_i^n u_1$ and $f^{n+1}(s) \to_i^n u_2$ and $u_1 \vDash \psi$ and $u_2 \nvDash \psi$. Since f^{n+1} is surjective (Prop. 6.4), we have $u_1 = f^{n+1}(t_1)$ and $u_2 = f^{n+1}(t_2)$ for some $t_1, t_2 \in S^{n+1}$, thus $f^{n+1}(s) \to_i^n f^{n+1}(t_1)$ and $f^{n+1}(s) \to_i^n f^{n+1}(t_2)$. By Prop. 6.5, there exist t_1', t_2' such that $s \to_i^{n+1} t_1'$ and $s \to_i^{n+1} t_2'$ and $f^{n+1}(t_1') = f^{n+1}(t_1)$ and $f^{n+1}(t_2') = f^{n+1}(t_2)$. From $u_1 \vDash \psi$ and $u_2 \nvDash \psi$ and IH, we can get $t_1' \vDash \psi$ and $t_2' \nvDash \psi$, respectively. $\mathcal{M}^{n+1}, s \nvDash \Delta_i \psi$. □

Define $f = f^1 \circ f^2 \circ \cdots \circ f^m$. By Prop. 6.4, it is easy to show that $f : S^m \to S^0$ is surjective. From Lemma 4.6 and Prop. 6.11, we have:

LEMMA 6.12. *For any $s \in S^m$ and any $\varphi \in$ **CL**, we have $\mathcal{M}^m, s \vDash \varphi$ iff $\varphi \in f(s)$.*

As every $u \in S^c = S^0$ is an image of some $s \in S^m$ under f, each maximal consistent set is satisfiable in $\mathcal{M}^m$, which implies the completeness theorem based on Prop. 6.8.

THEOREM 6.13 (Soundness and Completeness of $\mathbb{CLB}$). *$\mathbb{CLB}$ is sound and complete with respect to the class of symmetric frames.*

§7. Contingency logic with announcements.

In this section we add public announcement modalities to contingency logic. We will first give the language and its semantics, and then propose an axiomatization that can be shown to be complete because all formulas with announcements are provably equivalent to formulas without announcements (the proof system defines a rewrite procedure). We will give a case study of *muddy children puzzle* in this section, and a case study of *gossip protocols* in the next section, both in the 'knowing whether' setting, where the accessability relations are confined to equivalence relations.

DEFINITION 7.1 (Language **CLA**). *The language of **CLA** is obtained by adding an inductive clause $[\varphi]\varphi$ to the construction of the language **CL** (see Def. 2.1).*

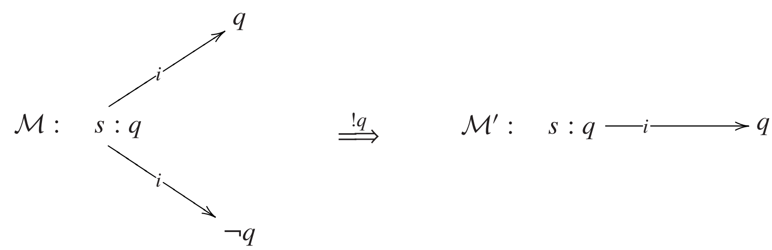
The formula $[\varphi]\psi$ says that "after every truthfully public announcement of φ, ψ holds".

DEFINITION 7.2. *Let $\mathcal{M} = \langle S, \{\to_i' \mid i \in \boldsymbol{I}\}, V\rangle$ be a model and $\varphi, \psi \in$ **CLA**. The semantics of public announcement is as follows.*

$$\mathcal{M}, s \vDash [\varphi]\psi \quad \Leftrightarrow \quad \mathcal{M}, s \vDash \varphi \text{ implies } \mathcal{M}|_\varphi, s \vDash \psi$$

where $\mathcal{M}|_\varphi = \langle S', \{\to_i' \mid i \in \boldsymbol{I}\}, V'\rangle$ is such that $S' = \{s \in S \mid \mathcal{M}, s \vDash \varphi\}$, $\to_i' = \to_i \cap (S' \times S')$, and $V'(p) = V(p) \cap S'$.

Unlike **CL**, the logic **CLA** is *not* closed under uniform substitution. For instance, $p \to [q]p$ is valid, but $\neg\Delta_i q \to [q]\neg\Delta_i q$ is not valid, as demonstrated by the following example, wherein $\mathcal{M}, s \not\models \neg\Delta_i q \to [q]\neg\Delta_i q$.



This is the reason why the proof system below must contain formula variables (schematic formulas) instead of propositional variables, and also for that reason we have presented the proof system $\mathbb{CL}$ in the same way.

DEFINITION 7.3 (Proof system $\mathbb{CLA}$). *The proof system $\mathbb{CLA}$ is the extension of $\mathbb{CL}$ (Def. 4.1) with the following reduction axioms for announcements.*[9]

!TOP	$[\varphi]\top \leftrightarrow \top$
!ATOM	$[\varphi]p \leftrightarrow (\varphi \to p)$
!NEG	$[\varphi]\neg\psi \leftrightarrow (\varphi \to \neg[\varphi]\psi)$
!CON	$[\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)$
!!	$[\varphi][\psi]\chi \leftrightarrow [\varphi \wedge [\varphi]\psi]\chi$
!Δ	$[\varphi]\Delta_i\psi \leftrightarrow (\varphi \to (\Delta_i[\varphi]\psi \vee \Delta_i[\varphi]\neg\psi))$

PROPOSITION 7.4 (Soundness). $\mathbb{CLA}$ *is sound with respect to the class of all frames.*

Proof. We only consider the nontrivial axiom schema $!\Delta$.

Left-to-right: Given any model $\mathcal{M} = \langle S, \{\to_i \mid i \in \mathbf{I}\}, V\rangle$ based on a frame and $s \in S$, assume that $\mathcal{M}, s \vDash [\varphi]\Delta_i\psi$. We now need to show that $\mathcal{M}, s \vDash \varphi \to (\Delta_i[\varphi]\psi \vee \Delta_i[\varphi]\neg\psi)$. For this, suppose $\mathcal{M}, s \vDash \varphi$, to show $\mathcal{M}, s \vDash \Delta_i[\varphi]\psi \vee \Delta_i[\varphi]\neg\psi$. By reductio ad absurdum we suppose $\mathcal{M}, s \nvDash \Delta_i[\varphi]\psi \vee \Delta_i[\varphi]\neg\psi$. Then $\mathcal{M}, s \nvDash \Delta_i[\varphi]\psi$ and $\mathcal{M}, s \nvDash \Delta_i[\varphi]\neg\psi$. That is to say, there exist $t, t' \in S$ such that $s \to_i t, s \to_i t'$ and $t \vDash [\varphi]\psi, t' \vDash \neg[\varphi]\psi$ and, there exist $u, u' \in S$ such that $s \to_i u, s \to_i u'$ and $u \vDash [\varphi]\neg\psi, u' \vDash \neg[\varphi]\neg\psi$. It follows that $\mathcal{M}, t' \vDash \varphi$ and $\mathcal{M}|_\varphi, t' \vDash \neg\psi$ from $t' \vDash \neg[\varphi]\psi$, and $\mathcal{M}, u' \vDash \varphi$ and $\mathcal{M}|_\varphi, u' \vDash \psi$ from $u' \vDash \neg[\varphi]\neg\psi$, where $\mathcal{M}|_\varphi$ is defined as Definition 7.2. Moreover, we have $s \to'_i t', s \to'_i u'$ in $\mathcal{M}|_\varphi$ because $\mathcal{M}, s \vDash \varphi, \mathcal{M}, t' \vDash \varphi, \mathcal{M}, u' \vDash \varphi$ and $s \to_i t', s \to_i u'$. Then $\mathcal{M}|_\varphi, s \nvDash \Delta_i\psi$, contradicting the assumption $\mathcal{M}, s \vDash [\varphi]\Delta_i\psi$ and $\mathcal{M}, s \vDash \varphi$.

Right-to-left: Assume $\mathcal{M}, s \vDash \varphi$. First consider the case that $\mathcal{M}, s \vDash \Delta_i[\varphi]\psi$. Then, either for all t with $s \to_i t$ we have $\mathcal{M}, t \vDash [\varphi]\psi$ or for all t with $s \to_i t$ we have $\mathcal{M}, t \vDash \neg[\varphi]\psi$. In the first case, with $\mathcal{M}, s \vDash \varphi$, we get for all t with $s \to'_i t$, $\mathcal{M}|_\varphi, t \vDash \psi$. In the second case, with $\mathcal{M}, s \vDash \varphi$, we get for all t with $s \to'_i t$, $\mathcal{M}|_\varphi, t \vDash \neg\psi$. In either subcase we both get $\mathcal{M}|_\varphi, s \vDash \Delta_i\psi$. Now consider the case that $\mathcal{M}, s \vDash \Delta_i[\varphi]\neg\psi$. Similarly, in this case we can also get $\mathcal{M}|_\varphi, s \vDash \Delta_i\psi$. Therefore we can conclude that $\mathcal{M}, s \vDash [\varphi]\Delta_i\psi$. □

The logic **CLA** is equally expressive as **CL**, as the axiomatization induces a rewrite procedure. By defining a suitable complexity, we can rewrite every formula in **CLA** as a logically equivalent formula of **CL** of lower complexity, and the rewriting will terminate

[9] We can replace axiom !! with the rule Sub for formulas of the form $[\varphi]\psi$; the similar argument goes for the axiom AA and formulas of the form $[\mathbf{M}, \mathbf{s}]\varphi$ in Def. 8.4. C.f. Wang & Cao (2013).

eventually, thus the completeness of $\mathbb{CLA}$ follows from that of $\mathbb{CL}$ (see van Ditmarsch *et al.*, 2007; Wang & Cao, 2013 for this reduction technique).

THEOREM 7.5 (Completeness of $\mathbb{CLA}$). *$\mathbb{CLA}$ is complete with respect to the class $\mathcal{K}$ of all frames.*

As axiomatization $\mathbb{CLA}$ gives a translation of **CLA** into **CL**, and **CL** is decidable (Prop. 4.8), the contingency logic with announcements is also decidable.

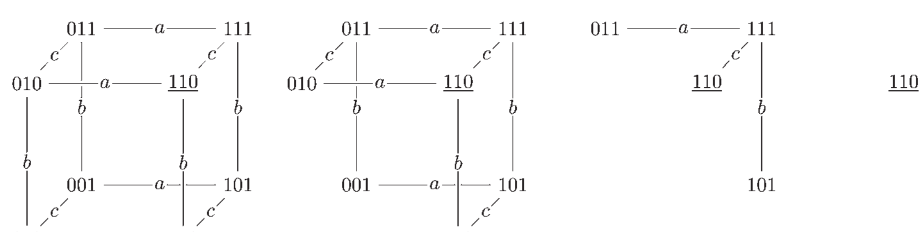
PROPOSITION 7.6. ***CLA** is decidable.*

We can also consider the contingency logic with announcements on other frame classes, where our main interest is the class of $\mathcal{S}5$ frames. The expressivity of contingency logics for other frame classes also does not change by adding the announcement operators, as the reduction axioms still allow every formula to be rewritten to an equivalent expression without announcements (so, a fortiori, this also holds for theorems of those logics).

THEOREM 7.7. *Consider the proof system $\mathbb{CLAS}5$ that extends $\mathbb{CLA}$ with ΔT and $w\Delta 5$. $\mathbb{CLAS}5$ is complete with respect to the class of $\mathcal{S}5$-frames.*

EXAMPLE 7.8. *The muddy children puzzle (c.f. e.g., Moses et al., 1986) can be formalized in terms of knowing whether. Children learn whether they are muddy by repeating the action of 'not stepping forward' which corresponds to the announcement 'nobody knows whether he is muddy'. Assume there are n children, of which $k \leq n$ are muddy. Let propositional variable m_i denote "i has mud on his forehead", let $\mathcal{M}_n$ be the model encoding the uncertainty, and let k be a world in that model where k children are muddy (child 1 to child k). The informative development for $n = 3$ and $k = 2$ is illustrated below. The announcement 'there is at least one muddy child' is formalized by $\bigvee_{i=1}^n m_i$ and 'nobody knows whether he is muddy' by $\bigwedge_{i=1}^n \nabla_i m_i$. We can now observe the following validities[10]—let for any φ, $[\varphi]^0\psi =_{df} \psi$ and $[\varphi]^{n+1}\psi =_{df} [\varphi][\varphi]^n\psi$, and w.l.o.g. assume that the first k children are muddy.*

$$\mathcal{M}_n, k \vDash [\bigvee_{i=1}^n m_i][\bigwedge_{i=1}^n \nabla_i m_i]^{k-1}\neg(\bigwedge_{i=1}^n \nabla_i m_i)$$
$$\mathcal{M}_n, k \vDash [\bigvee_{i=1}^n m_i][\bigwedge_{i=1}^n \nabla_i m_i]^{k-1}(\bigwedge_{i=1}^k \Delta_i m_i \wedge \bigwedge_{i=k+1}^n \nabla_i m_i)$$
$$\mathcal{M}_n, k \vDash [\bigvee_{i=1}^n m_i][\bigwedge_{i=1}^n \nabla_i m_i]^{k-1}[\bigwedge_{i=1}^k \Delta_i m_i] \bigwedge_{i=k+1}^n \Delta_i m_i$$



§8. Contingency logic with action models. In this section we extend contingency logic with public announcements to contingency logic with more general forms of

10 Note that the higher-order 'knowing whether' is hidden in the announcements: when the formulas with announcements are translated into formulas without announcements, there will be higher-order 'knowing whether', such as '*a* knows whether *b* knows whether m_b'.

information change called *action models* (Baltag *et al.*, 1998). We will first give the language and its semantics, and then propose an axiomatization, which indicates the obtained logic is equally expressive as contingency logic. We denote the language of contingency logic with action models by **CLAM**.

DEFINITION 8.1 (Language **CLAM**). *The language of **CLAM** is defined recursively by:*

$$\begin{array}{rl} \varphi ::= & \top \mid p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Delta_i \varphi \mid [\alpha]\varphi \\ \alpha ::= & (\mathbf{M}, \mathbf{s}) \mid (\alpha \cup \alpha) \end{array}$$

*Where $p \in \boldsymbol{P}$, $i \in \boldsymbol{I}$, and $(\mathbf{M}, \mathbf{s})$ is a pointed action model, where $\mathbf{M} = \langle \mathbf{S}, \{\rightarrow_{\mathbf{i}} \mid \mathbf{i} \in \boldsymbol{I}\}, \mathbf{pre}\rangle$ with $s \in \mathbf{S}$ and $\mathbf{S}$ is a finite set of action points, for all $\mathbf{i} \in \boldsymbol{I}$, $\rightarrow_{\mathbf{i}}$ is a binary relation on $\mathbf{S}$, and $\mathbf{pre}$ is a precondition function from $\mathbf{S}$ to **CLAM**.*

The formula $[\alpha]\varphi$ says that "after every execution of the action α, φ holds".

DEFINITION 8.2 (Composition of action models). *Given two action models $\mathbf{M} = \langle \mathbf{S}, \{\rightarrow_{\mathbf{i}} \mid \mathbf{i} \in \boldsymbol{I}\}, \mathbf{pre}\rangle$ and $\mathbf{M}' = \langle \mathbf{S}', \{\rightarrow'_{\mathbf{i}} \mid \mathbf{i} \in \boldsymbol{I}\}, \mathbf{pre}'\rangle$. Then their* composition $(\mathbf{M}; \mathbf{M}')$ *is the action model*
$\langle \mathbf{S}'', \{\rightarrow''_{\mathbf{i}} \mid \mathbf{i} \in \boldsymbol{I}\}, \mathbf{pre}''\rangle$ *where*

$$\begin{array}{lcl} \mathbf{S}'' & = & \mathbf{S} \times \mathbf{S}' \\ (\mathbf{s}, \mathbf{s}') \rightarrow''_{\mathbf{i}} (\mathbf{t}, \mathbf{t}') & \Leftrightarrow & \mathbf{s} \rightarrow_{\mathbf{i}} \mathbf{t} \text{ and } \mathbf{s}' \rightarrow'_{\mathbf{i}} \mathbf{t}' \\ \mathbf{pre}''(\mathbf{s}, \mathbf{s}') & = & \langle \mathbf{M}, \mathbf{s}\rangle \mathbf{pre}(\mathbf{s}') \end{array}$$

DEFINITION 8.3. *Given a Kripke model $\mathcal{M} = \langle S, \{\rightarrow_i \mid i \in \boldsymbol{I}\}, V\rangle$, an action model $\mathbf{M} = \langle \mathbf{S}, \{\rightarrow_{\mathbf{i}} \mid \mathbf{i} \in \boldsymbol{I}\}, \mathbf{pre}\rangle$, a formula $\varphi \in$ **CLAM**, and an action α. The semantics of **CLAM** is as follows (we only consider the new cases):*

$$\begin{array}{lcl} \mathcal{M}, s \vDash [\alpha]\varphi & \Leftrightarrow & \text{for all } (\mathcal{M}', s') : (\mathcal{M}, s) \overset{\alpha}{\Longrightarrow} (\mathcal{M}', s') \text{ implies } \mathcal{M}', s' \vDash \varphi \\ (\mathcal{M}, s) \overset{\mathbf{M},\mathbf{s}}{\Longrightarrow} (\mathcal{M}', s') & \Leftrightarrow & \mathcal{M}, s \vDash \mathbf{pre}(\mathbf{s}) \text{ and } (\mathcal{M}', s') = (\mathcal{M} \otimes \mathbf{M}, (s, \mathbf{s})) \\ \overset{\alpha \cup \alpha'}{\Longrightarrow} & = & \overset{\alpha}{\Longrightarrow} \cup \overset{\alpha'}{\Longrightarrow} \end{array}$$

where $\mathcal{M}' = \mathcal{M} \otimes \mathbf{M}$ is the update product of Kripke model $\mathcal{M}$ and action model $\mathbf{M}$, defined as $\langle S', \{\rightarrow'_i \mid i \in \boldsymbol{I}\}, V'\rangle$ with

$$\begin{array}{lcl} S' & = & \{(s, \mathbf{s}) \mid s \in S, \mathbf{s} \in \mathbf{S} \text{ and } \mathcal{M}, s \vDash \mathbf{pre}(\mathbf{s})\} \\ (s, \mathbf{s}) \rightarrow'_i (t, \mathbf{t}) & \Leftrightarrow & s \rightarrow_i t \text{ and } \mathbf{s} \rightarrow_{\mathbf{i}} \mathbf{t} \\ (s, \mathbf{s}) \in V'(p) & \Leftrightarrow & s \in V(p), \text{ for all } p \in \boldsymbol{P} \end{array}$$

DEFINITION 8.4 (Proof system $\mathbb{CLAM}$). *The proof system $\mathbb{CLAM}$ is the extension of $\mathbb{CL}$ with the following reduction axioms for action models.*

ATOP	$[\mathbf{M}, \mathbf{s}]\top \leftrightarrow \top$
AATOM	$[\mathbf{M}, \mathbf{s}]p \leftrightarrow (\mathbf{pre}(\mathbf{s}) \rightarrow p)$
ANEG	$[\mathbf{M}, \mathbf{s}]\neg\varphi \leftrightarrow (\mathbf{pre}(\mathbf{s}) \rightarrow \neg[\mathbf{M}, \mathbf{s}]\varphi)$
ACON	$[\mathbf{M}, \mathbf{s}](\varphi \wedge \psi) \leftrightarrow ([\mathbf{M}, \mathbf{s}]\varphi \wedge [\mathbf{M}, \mathbf{s}]\psi)$
AΔ	$[\mathbf{M}, \mathbf{s}]\Delta_i \psi \leftrightarrow (\mathbf{pre}(\mathbf{s}) \rightarrow \bigwedge_{\mathbf{s} \rightarrow_{\mathbf{i}} \mathbf{t}} (\Delta_i [\mathbf{M}, \mathbf{t}]\psi \vee \Delta_i [\mathbf{M}, \mathbf{t}]\neg\psi))$
AA	$[\mathbf{M}, \mathbf{s}][\mathbf{M}', \mathbf{s}']\varphi \leftrightarrow [(\mathbf{M}, \mathbf{s}); (\mathbf{M}', \mathbf{s}')]\varphi$
AC	$[\alpha \cup \beta]\varphi \leftrightarrow ([\alpha]\varphi \wedge [\beta]\varphi)$

PROPOSITION 8.5 (Soundness). $\mathbb{CLAM}$ *is sound with respect to the class $\mathcal{K}$ of all frames.*

Proof. We only need to consider the nontrivial axiom schema $\mathtt{A\Delta}$. Given any model $\mathcal{M} = \langle S, \{\to_i \mid i \in \mathbf{I}\}, V\rangle$ and $s \in S$.

Left-to-right: assume that $\mathcal{M}, s \vDash [\mathbf{M}, \mathbf{s}]\Delta_i\psi$. We only need to show $\mathcal{M}, s \vDash \mathbf{pre}(\mathbf{s}) \to \bigwedge_{\mathbf{s}\to_{\mathbf{i}}\mathbf{t}}(\Delta_i[\mathbf{M}, \mathbf{t}]\psi \vee \Delta_i[\mathbf{M}, \mathbf{t}]\neg\psi)$. For this, suppose $\mathcal{M}, s \vDash \mathbf{pre}(\mathbf{s})$, to show $\mathcal{M}, s \vDash \bigwedge_{\mathbf{s}\to_{\mathbf{i}}\mathbf{t}}(\Delta_i[\mathbf{M}, \mathbf{t}]\psi \vee \Delta_i[\mathbf{M}, \mathbf{t}]\neg\psi)$. By reductio and absurdum we suppose that for some $\mathbf{t}$ such that $\mathbf{s} \to_{\mathbf{i}} \mathbf{t}$ and $\mathcal{M}, s \nvDash \Delta_i[\mathbf{M}, \mathbf{t}]\psi \vee \Delta_i[\mathbf{M}, \mathbf{t}]\neg\psi$. That is to say, there exist $t, t' \in S$ such that $s \to_i t$, $s \to_i t'$ and $t \vDash [\mathbf{M}, \mathbf{t}]\psi$, $t' \vDash \neg[\mathbf{M}, \mathbf{t}]\psi$ and, there exist $u, u' \in S$ such that $s \to_i u$, $s \to_i u'$ and $u \vDash [\mathbf{M}, \mathbf{t}]\neg\psi$, $u' \vDash \neg[\mathbf{M}, \mathbf{t}]\neg\psi$. It follows that $\mathcal{M}, t' \vDash \mathbf{pre}(\mathbf{t})$ and $(\mathcal{M} \otimes \mathbf{M}, (t', \mathbf{t})) \vDash \neg\psi$ from $t' \vDash \neg[\mathbf{M}, \mathbf{t}]\psi$, and $\mathcal{M}, u' \vDash \mathbf{pre}(\mathbf{t})$ and $(\mathcal{M} \otimes \mathbf{M}, (u', \mathbf{t})) \vDash \psi$ from $u' \vDash \neg[\mathbf{M}, \mathbf{t}]\neg\psi$. By $\mathbf{s} \to_{\mathbf{i}} \mathbf{t}$ and $s \to_i t'$ and $s \to_i u'$ we get $(s, \mathbf{s}) \to_i' (t', \mathbf{t})$ and $(s, \mathbf{s}) \to_i' (u', \mathbf{t})$, and then $(\mathcal{M} \otimes \mathbf{M}, (s, \mathbf{s})) \nvDash \Delta_i\psi$, contracting to $\mathcal{M}, s \vDash [\mathbf{M}, \mathbf{s}]\Delta_i\psi$ and $\mathcal{M}, s \vDash \mathbf{pre}(\mathbf{s})$.

Right-to-left: assume that $\mathcal{M}, s \vDash \mathbf{pre}(\mathbf{s})$ and let $\mathbf{t}$ satisfy $\mathbf{s} \to_{\mathbf{i}} \mathbf{t}$. First consider the case that $\mathcal{M}, s \vDash \Delta_i[\mathbf{M}, \mathbf{t}]\psi$. Then either for all t with $s \to_i t$ we have $\mathcal{M}, t \vDash [\mathbf{M}, \mathbf{t}]\psi$ or for all t with $s \to_i t$ we have $\mathcal{M}, t \vDash \neg[\mathbf{M}, \mathbf{t}]\psi$. In the first case, we get for all t such that $s \to_i t$ and $\mathcal{M}, t \vDash \mathbf{pre}(\mathbf{t})$, $(\mathcal{M} \otimes \mathbf{M}, (t, \mathbf{t})) \vDash \psi$, thus we have for all $(t, \mathbf{t})$: $(s, \mathbf{s}) \to_i' (t, \mathbf{t})$ implies $(\mathcal{M} \otimes \mathbf{M}, (t, \mathbf{t})) \vDash \psi$. In the second case, we get for all $(t, \mathbf{t})$: $(s, \mathbf{s}) \to_i' (t, \mathbf{t})$ implies $(\mathcal{M} \otimes \mathbf{M}, (t, \mathbf{t})) \vDash \neg\psi$. In either subcase we get $(\mathcal{M} \otimes \mathbf{M}, (s, \mathbf{s})) \vDash \Delta_i\psi$. Now consider the case that $\mathcal{M}, s \vDash \Delta_i[\mathbf{M}, \mathbf{t}]\neg\psi$. Similarly, in this case we can get $(\mathcal{M} \otimes \mathbf{M}, (s, \mathbf{s})) \vDash \Delta_i\psi$. Then we can conclude that $\mathcal{M}, s \vDash [\mathbf{M}, \mathbf{s}]\Delta_i\psi$. □

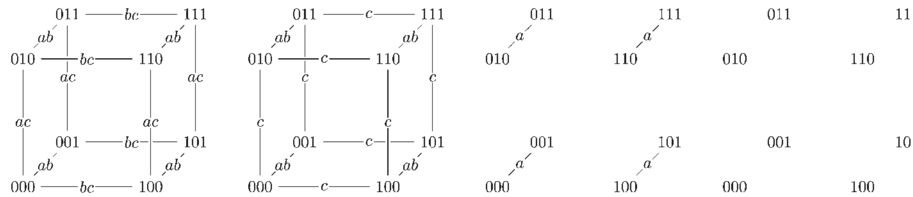
Similar to the logic **CLA**, **CLAM** is equally expressive as **CL**, as the axiomatization induces a rewrite procedure. By defining a suitable complexity, we can rewrite every formula in **CLAM** as a logically equivalent formula of **CL** of lower complexity, and the rewriting will terminate eventually, thus we get the completeness of $\mathbb{CLAM}$ from that of $\mathbb{CL}$.

THEOREM 8.6 (Completeness of $\mathbb{CLAM}$). $\mathbb{CLAM}$ *is complete with respect to the class* $\mathcal{K}$ *of all frames.*

Similar to the analysis preceding Thm. 7.7, we can obtain the complete axiomatizations on other frame classes, especially on $\mathcal{S}5$. Due to the fact that **CLAM** is equally expressive as **CL**, and **CL** is decidable, we have the logics **CLAM**, **CLA** and **CL** are all equally expressive, and

PROPOSITION 8.7. *Contingency logic with action models **CLAM** is decidable.*

EXAMPLE 8.8. *In epistemic logical treatments of* gossip protocols *discussed by Wang et al. (2011) and Attamah et al. (2014), agents exchange information by telephone calls. We can assume calls ij between agents i and j are calls of type ab^- in Attamah et al. (2014). Initially, every agent only knows (the value of) a single secret, and agents exchange all the secrets they know when they call each other, and noncallers can observe the callers but not hear the secrets being exchanged. We can also assume such secrets are propositional, that is, they have binary values* (0 *and* 1). *For three agents a, b, c with secrets m_a, m_b and m_c, a sequence to distribute all secrets is ab; bc; ac, which makes all agents know all secrets. The initial model $\mathcal{M}^n$ of uncertainty is much like the initial model of uncertainty for muddy children: in muddy children you only know the secrets (muddy or not muddy) of everybody else, in gossip you only know your own secret. We can now say that $\mathcal{M}^3 \models [ab; bc; ac] \bigwedge_{i\in\{a,b,c\}} \bigwedge_{j\in\{a,b,c\}} \Delta_i m_j$. Below we depict the informative consequences of the call sequence ab; bc; ac in the initial model $\mathcal{M}^3$. The action model of the 1st call ab has four points, with preconditions for the four valuations of m_a and m_b, with identity* *relation for a, b and the universal relation for c. The action models of 2nd and 3rd calls have both eight points, with preconditions for the eight valuations of m_a, m_b, and m_c.*



§9. Comparison with the literature. As mentioned in the introduction, Humberstone (1995), Kuhn (1995), and Zolin (1999) successively provide axiomatizations for contingency logic on various frame classes and prove the completeness, except for symmetric frames. Humberstone (1995) provides an infinitary axiomatization **NC** over arbitrary frames, with infinitely many rules $(\text{NCR})_k$. This system is simplified as $\mathbf{K}\Delta$ in Kuhn (1995).[11] Zolin (1999) modifies $\mathbf{K}\Delta$ to make it similar to the minimal modal logic. It is a must to compare our axiomatizations and proof method with the literature on contingency logic and the logic of ignorance.

To show the completeness, the existing work all adopt the canonical model construction method, where the key part is to define a suitable canonical relation. In order to simulate the canonical relation in modal logic, Humberstone defines a complicated function λ from maximal consistent sets to the subsets of **CL**, by $\lambda(s) = \{\varphi \mid \Delta\varphi \in s$ and for all $\psi, \vdash \varphi \to \psi$ implies $\Delta\psi \in s\}$, which is responsible for the infinitary axiomatization, and the canonical relation is defined by $sR^c t$ just in case $\lambda(s) \subseteq t$. The rather complicated proof also requires König's Lemma. Kuhn ingeniously simplifies the definition of λ, by setting $\lambda(s) = \{\varphi \mid \text{ for every } \psi, \Delta(\varphi \vee \psi) \in s\}$, which makes the completeness proof much easier than the method used in Humberstone (1995). Zolin defines a function $\sharp$ from maximal consistent sets to the subsets of **CL**, as $\sharp(s) = \{\varphi \mid \boxtimes\varphi \subseteq s\}$, where $\boxtimes\varphi = \{\Delta(\psi \to \varphi) \mid \psi \in \mathbf{CL}\}$. Note that Kuhn's λ and Zolin's $\sharp$ are the same function, in the sense that for all maximal consistent sets s, $\sharp(s) = \lambda(s)$, as one can show.

Our system $\mathbb{CL}$ is closest to Kuhn's $\mathbf{K}\Delta$, except that our axiom $\Delta\texttt{Con}$ differs from axiom $\Delta\varphi \wedge \nabla(\varphi \wedge \psi) \to \nabla\psi$ there. However, our proof method is based on the almost-definability schema AD (Prop. 3.5), which is very different from the methods of Kuhn (Humberstone, Zolin and other researchers). The idea of AD also inspired a notion of bisimulation for **CL** (called 'Δ-bisimulation'), which is used to characterize the expressive power of **CL** within modal logic and within first-order logic; more precisely, a modal formula (resp. a first-order formula) is equivalent to a **CL**-formula iff it is invariant under Δ-bisimulation (Fan *et al.*, 2014). Kuhn's method has its limitations, since the necessity operator, defined by $\boxtimes\varphi =_{df} \bigwedge_{\psi \in \mathbf{CL}} \Delta(\varphi \vee \psi)$, is not really $\Box$. For instance, $\varphi \to \boxtimes\neg\boxtimes\neg\varphi$ is not valid on the class of symmetric frames, as observed in Zolin (2001). Besides, as Humberstone (2002, p. 118) questioned, the canonical relations in Kuhn (1995) and Zolin (1999) at least do not apply to the reflexive frames, a fortiori, they do not apply to the symmetric frames. Comparatively, we really define necessity in terms of contingency in the general sense, and our method can work for all the usual frame properties in a rather uniform fashion, among which the case for symmetric axiomatization is highly nontrivial.

[11] As pointed out by George Schumm in the review of Kuhn (1995), Kuhn miswrote $\mathbf{K}\Delta$ on p. 231 as $\mathbf{K4}\Delta$.

Moreover, we extend the results to public announcements and action models, which were not discussed in the literature of contingency logic.

Apparently unaware of the literature of contingency logic, van der Hoek & Lomuscio (2003, 2004) give a complete axiomatization of a logic of ignorance with primitive modal construct $I\varphi$, for 'the agent is ignorant about φ'. If an agent is ignorant about φ, she does not know whether φ, so $I\varphi$ is definable as $\neg\Delta\varphi$. Their axiomatization **Ig** is shown in Def. 9.1, wherein we have replaced I by $\neg\Delta_i$. It is different from ours. Now it is of course a matter of taste whether one prefers the system $\mathbb{CL}$ (page 82) or the one below, but we tend to find ours simpler, e.g. with respect to the axioms `I3` and `I4` below. Although motivated in an epistemic setting, van der Hoek & Lomuscio (2003, 2004) axiomatized **CL** over the class of arbitrary frames.

DEFINITION 9.1 (Axiomatization **Ig**, van der Hoek & Lomuscio (2003, 2004)).

`I0`	*All instances of propositional tautologies*
`I1`	$\neg\Delta_i\varphi \leftrightarrow \neg\Delta_i\neg\varphi$
`I2`	$\neg\Delta_i(\varphi\wedge\psi) \to \neg\Delta_i\varphi \vee \neg\Delta_i\psi$
`I3`	$(\Delta_i\varphi \wedge \neg\Delta_i(\chi_1\wedge\varphi) \wedge \Delta_i(\varphi\to\psi) \wedge \neg\Delta_i(\chi_2 \wedge (\varphi\to\psi)))$ $\to \Delta_i\psi \wedge \neg\Delta_i(\chi_1\wedge\psi)$
`I4`	$\Delta_i\psi \wedge \neg\Delta_i\chi \to \neg\Delta_i(\chi\wedge\psi) \vee \neg\Delta_i(\chi\wedge\neg\psi)$
`RI`	*From* φ *infer* $\Delta_i\varphi \wedge (\neg\Delta_i\chi \to \neg\Delta_i(\chi\wedge\varphi))$
`MP`	*Modus Ponens*
`Sub`	*Substitution of equivalents*

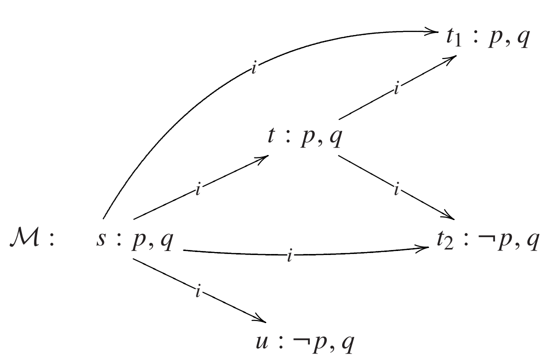
Since both systems are complete, their axioms and inference rules are interderivable with ours, while the details of the direct proofs of the interderivability are omitted.

PROPOSITION 9.2. *All the axioms of* **Ig** *are provable in* $\mathbb{CL}$ *and all the rules of* **Ig** *are admissible in* $\mathbb{CL}$, *and vice versa.*

The proof system **Ig** is also extended with an axiom `G4`, which we present in terms of Δ_i:

$$\neg\Delta_i\chi \to \big(\Delta_i\varphi\wedge\neg\Delta_i(\varphi\wedge\chi) \to \Delta_i(\Delta_i\varphi\wedge\neg\Delta_i(\chi\wedge\varphi))\wedge\neg\Delta_i(\Delta_i\varphi\wedge\neg\Delta_i(\varphi\wedge\chi)\wedge\chi)\big)$$

It is then claimed that **Ig** + `G4` is a complete axiomatization of the logic of ignorance over transitive frames (van der Hoek & Lomuscio, 2004, Lem. 4.2). Unfortunately, `G4` is invalid, thus the system is not sound. Consider this countermodel $\mathcal{M}$



and the formula

$$\neg\Delta_i p \to (\Delta_i q\wedge\neg\Delta_i(q\wedge p) \to \Delta_i(\Delta_i q\wedge\neg\Delta_i(p\wedge q))\wedge\neg\Delta_i(\Delta_i q\wedge\neg\Delta_i(q\wedge p)\wedge p))$$

Observe $s \vDash \neg\Delta_i p$ and $s \vDash \Delta_i q \wedge \neg\Delta_i(q \wedge p)$. Then, note that $s \nvDash \Delta_i(\Delta_i q \wedge \neg\Delta_i(p \wedge q))$ (take u and t as two witnesses), thus $s \nvDash \Delta_i(\Delta_i q \wedge \neg\Delta_i(p \wedge q)) \wedge \neg\Delta_i(\Delta_i q \wedge \neg\Delta_i(q \wedge p) \wedge p)$. Therefore, this formula is *false* in state s of this model $\mathcal{M}$, which invalidates G4.[12]

In this paper, we advanced the research beyond van der Hoek & Lomuscio (2004) by proving expressivity results and more undefinability results. And more importantly, apart from correctly axiomatizing contingency logic over transitive frames (the system $\mathbb{CL}4$), we also axiomatized **CL** on various other frame classes, which was considered hard in van der Hoek & Lomuscio (2004). Further, we extended contingency logic with public announcements and with action models, and gave complete axiomatization for these extensions.

Another recent work on a logic of ignorance is Steinsvold (2008). The author gives a topological semantics for the logic of ignorance and completely axiomatizes it by the following proof system **LB** (we have replaced the nonstandard notation $\Box$ in Steinsvold (2008) by Δ_i):

DEFINITION 9.3 (Axiomatization **LB**).

TAUT	*All instances of propositional tautologies*
N	$\Delta_i \top \leftrightarrow \top$
Z	$\Delta_i \varphi \leftrightarrow \Delta_i \neg\varphi$
R	$\Delta_i \varphi \wedge \Delta_i \psi \to \Delta_i(\varphi \wedge \psi)$
WM	*From* $\Delta_i\varphi \wedge \varphi \to \psi$ *infer* $\Delta_i\varphi \wedge \varphi \to \Delta_i\psi \wedge \psi$
MP	*Modus Ponens*
Sub	*Substitution of equivalents*

This proof system is equivalent to our system $\mathbb{CLS}4$ for **CL** over $\mathcal{S}4$-frames in the following sense.

PROPOSITION 9.4. *All the axioms of* **LB** *are provable in* $\mathbb{CLS}4$ *and all the rules of* **LB** *are admissible in* $\mathbb{CLS}4$*, and vise versa.*

Proof. We show that WM is admissible in $\mathbb{CLS}4$.[13] Other proofs are omitted.

Suppose that $\vdash \Delta_i\varphi \wedge \varphi \to \psi$, we need to show that $\vdash \Delta_i\varphi \wedge \varphi \to \Delta_i\psi \wedge \psi$. By supposition and NEC$\Delta$, we have $\vdash \Delta_i(\Delta_i\varphi \wedge \varphi \to \psi)$. Using ΔT, we can get $\vdash (\Delta_i\varphi \wedge \varphi) \wedge \Delta_i(\Delta_i\varphi \wedge \varphi) \to \Delta_i\psi$. Moreover, we can show $\vdash \Delta_i\varphi \wedge \varphi \to \Delta_i(\Delta_i\varphi \wedge \varphi)$ as follows: by TAUT, we obtain $\vdash \Delta_i\varphi \to (\varphi \to \Delta_i\varphi \wedge \varphi)$, then NEC$\Delta$ implies $\vdash \Delta_i(\Delta_i\varphi \to (\varphi \to \Delta_i\varphi \wedge \varphi))$. Using ΔT twice, we can get $\vdash \Delta_i\varphi \wedge \Delta_i\Delta_i\varphi \wedge \varphi \wedge \Delta_i\varphi \to \Delta_i(\Delta_i\varphi \wedge \varphi)$. From w$\Delta$4 it follows that $\vdash \Delta_i\varphi \to \Delta_i\Delta_i\varphi$, thus $\vdash \Delta_i\varphi \wedge \varphi \to \Delta_i(\Delta_i\varphi \wedge \varphi)$, and then $\vdash \Delta_i\varphi \wedge \varphi \to \Delta_i\psi$. By supposition again, we conclude that $\vdash \Delta_i\varphi \wedge \varphi \to \Delta_i\psi \wedge \psi$, as desired. $\Box$

Unlike Prop. 9.2, Prop. 9.4 cannot be obtained using the completeness of both systems, since the semantics of the two logics are different. Compared to $\mathbb{CLS}4$, the axioms of **LB** are simpler, while the rules are more complicated (WM is clearly a complex derivation rule, and in $\mathbb{CLS}4$ the rule Sub is admissible instead). It is again a matter of taste which system is preferable. Nevertheless, the above result also shows that the topological semantics in Steinsvold (2008) is equivalent to our Kripke semantics over $\mathcal{S}4$-frames, modulo validity.

---

12 Confirmed by the authors of van der Hoek & Lomuscio (2004) by personal communication.

13 Here again, by abuse of notation, we use $\vdash$ to denote $\vdash_{\mathbb{CLS}4}$.

But note that the axiomatization and completeness results in Steinsvold (2008) seem *unable* to apply to or be easily adapted to the weaker systems than $\mathbb{CLS}4$, since $\mathtt{WM}$ is crucial to obtain a topology.

§10. Conclusions and further research. In this paper, we showed that necessity is almost definable in terms of contingency, which is demonstrated by the almost-definability schema AD: $\nabla_i \chi \to (\Box_i \varphi \leftrightarrow \Delta_i \varphi \wedge \Delta_i (\chi \to \varphi))$. Inspired by the schema, we axiomatized the contingency logic over various frames, via an intuitive and rather uniform method. The axiomatization for (multi-agent) $\mathcal{B}$ and its completeness proof, which are missing in the literature, are highly nontrivial. The other axiomatizations and their proofs are different from the existing ones in literature. We also axiomatized contingency logic with public announcements and action models, and demonstrated that the two logics are both equally expressive as **CL**. We also illustrated that, by only using the weaker operator 'knowing whether' rather than 'knowing that', muddy children puzzle and gossip protocols can be expressed more succinctly and naturally.

We compared our work to the literature on contingency logic and the literature on the logic of ignorance. As for the completeness proof methods, we argued that our almost-definability-based method is better than the existing methods on contingency logic, in the sense that it is more intuitive and, more importantly, it can deal with all the usual frame classes, which does not hold for the methods in the literature. Among our results, we characterized a logic of opinionatedness (Thm. 5.13) and a logic of knowing whether (Thm. 5.15), where $\Delta_i \varphi$ are read 'i is opinionated as to whether φ' and 'i knows whether φ', respectively. We also discussed the literature on ignorance logic, showed the interderivability between our $\mathbb{CL}$ and **Ig**, and between $\mathbb{CLS}4$ and **LB**, proved that the proof system **Ig** $+$ $\mathtt{G4}$ in van der Hoek & Lomuscio (2004) is not sound over transitive frames, and we gave a correct axiomatization for it.

We systematically compared the expressivity between **CL** and **ML**: **CL** is less expressive than **ML** over model classes $\mathcal{K}, \mathcal{D}, \mathcal{B}$, 4, and 5. It is equally expressive as **ML** over $\mathcal{T}$ (and classes contained in $\mathcal{T}$, such as $\mathcal{S}4$ and $\mathcal{S}5$). Besides, by using methods different from the ones in the literature, we proved that the frame classes $\mathcal{D}, \mathcal{T}, \mathcal{B}$, 4, and 5 are not definable in **CL**, and showed that $\mathbb{CL}+\mathtt{w}\Delta 4$ and $\mathbb{CL}+\mathtt{w}\Delta 5$ are incomplete w.r.t. transitive frames and Euclidean frames, respectively.

There are a lot of directions for further research. Here we list some of them.

- As mentioned in the introduction, 'knowing whether' seems to be a natural modality which can express things succinctly. It is shown in van Ditmarsch *et al.* (2014) that **CL** is exponentially more succinct than **ML** on $\mathcal{K}$. We conjecture that **CL** over $\mathcal{S}5$ is also exponentially more succinct than **ML** if there are at least two agents. The computational complexity of contingency logics is also left for future work.
- The comparison with Steinsvold (2008) demonstrates that the same logic may be obtained by different semantics based on different models. The undefinability of frame properties suggests that the Kripke semantics may not be the best semantics for contingency logic. We intend to investigate neighbourhood semantics and other weaker semantics for **CL**.
- We may consider adding group operators for knowing-whether (or ignorance) to the language. There are various options to define such group operators. Is a group G ignorant of φ if, when defining the accessibility relation for G as the transitive closure of the union of all relations, both a state where φ is true and a state where

φ is false are group-accessible? Or should all agents consider states possible where φ is true and where φ is false, and then we 'simply' take Kleene-iteration of that? There are yet other ways to define group ignorance, and the notion of group ignorance is under close scrutiny in formal epistemology (Hansen, 2011; Hendricks, 2010).

- We may consider adding arbitrary announcement operators discussed by Balbiani *et al.* (2008) to knowing-whether logic. One can then express, for example, that after any announcement agent i remains ignorant. This addition becomes more challenging if one then removes the announcement operators from the logical language and defines the arbitrary announcement by modally definable model restrictions.
- We may combine knowing-whether logic with planning. One can investigate one planning called '*knowing-whether planning*', where the goal formula is of the form $\Delta_i\varphi$, expressing that agent i knows whether φ, rather than stronger $\Box_i\varphi$, which expresses that agent i knows that φ. Similar idea is mentioned in Petrick & Bacchus (2004).
- There are other interesting epistemic modalities beyond 'knowing that', such as 'knowing what', 'knowing who', 'knowing how' and so on. In particular, 'knowing whether' can be viewed as a special case of 'knowing what': knowing whether φ is knowing *what* the truth value of φ is. Such new operators deserve further logical investigations (cf. Wang & Fan, 2013, 2014).

§11. Acknowledgements. We are grateful to Philippe Balbiani for the pointers to the literature on contingency logic. We also thank the anonymous reviewers for their insightful comments. Jie Fan gratefully acknowledges the support from China Scholarship Council. Yanjing Wang acknowledges support of the key project 12&ZD119 of National Social Science Foundation of China. Hans van Ditmarsch is also affiliated to IMSc (Institute of Mathematical Sciences), Chennai, as research associate. He acknowledges supports from European Research Council grant EPS 313360.

BIBLIOGRAPHY

Aloni, M., Égré, P., & Jager, T. (2013). Knowing whether A or B. *Synthese*, **190**(14), 2595–2621.

Attamah, M., van Ditmarsch, H., Grossi, D., & van der Hoek, W. (2014). Knowledge and gossip. In *Proceedings of 21st ECAI*. Amsterdam: IOS Press, pp. 21–26.

Balbiani, P., Baltag, A., van Ditmarsch, H., Herzig, A., Hoshi, T., & De Lima, T. (2008). 'Knowable' as 'known after an announcement'. *Review of Symbolic Logic*, **1**(3), 305–334.

Baltag, A., Moss, L. S., & Solecki, S. (1998). The logic of public announcements, common knowledge, and private suspicions. In *Proceedings of the 7th TARK*, San Francisco, CA: Morgan Kaufmann, pp. 43–56.

Brogan, A. (1967). Aristotle's logic of statements about contingency. *Mind*, **76**(301), 49–61.

Ciardelli, I., & Roelofsen, F. (2014). Inquisitive dynamic epistemic logic. *Synthese*, doi:10.1007/s11229-014-0404-7. To appear.

Demri, S. (1997). A completeness proof for a logic with an alternative necessity operator. *Studia Logica*, **58**(1), 99–112.

Égré, P. (2008). Question-embedding and factivity. *Grazer Philosophische Studien*, **77**, 85–125.

Fan, J., Wang, Y., & van Ditmarsch, H. (2014). Almost necessary. *Advances in Modal Logic*, Vol. 10. pp. 178–196.

Hansen, J. (2011). A logic toolbox for modeling knowledge and information in multi-agent systems and social epistemology. PhD Thesis, Roskilde University.

Hart, S., Heifetz, A., & Samet, D. (1996). Knowing whether, knowing that, and the cardinality of state spaces. *Journal of Economic Theory*, **70**(1), 249–256.

Hedetniemi, S., Hedetniemi, S., & Liestman, A. (1988). A survey of gossiping and broadcasting in communication networks. *Networks*, **18**, 319–349.

Heifetz, A., & Samet, D. (1993). *Universal Partition Structures*. Working paper (Israel Institute of business research). Tel Aviv: Tel Aviv University.

Hendricks, V. (2010). Knowledge transmissibility and pluralistic ignorance: A first stab. *Metaphilosophy*, **41**, 279–291.

Humberstone, L. (1995). The logic of noncontingency. *Notre Dame Journal of Formal Logic*, **36**(2), 214–229.

Humberstone, L. (2002). The modal logic of agreement and noncontingency. *Notre Dame Journal of Formal Logic*, **43**(2), 95–127.

Kuhn, S. (1995). Minimal non-contingency logic. *Notre Dame Journal of Formal Logic*, **36**(2), 230–234.

McCarthy, J. (1979). First-order theories of individual concepts and propositions. *Machine Intelligence*, **9**, 129–147.

Montgomery, H., & Routley, R. (1966). Contingency and non-contingency bases for normal modal logics. *Logique et Analyse*, **9**, 318–328.

Moses, Y., Dolev, D., & Halpern, J. (1986). Cheating husbands and other stories: a case study in knowledge, action, and communication. *Distributed Computing*, **1**(3), 167–176.

Petrick, R., & Bacchus, F. (2004). Extending the knowledge-based approach to planning with incomplete information and sensing. In *Proceedings of the 9th KR*, Palo Alto: AAAI Press, pp. 613–622.

Plaza, J. (1989). Logics of public communications. In *Proceedings of the 4th ISMIS*. Oak Ridge, TN: Oak Ridge National Laboratory, pp. 201–216.

Reiter, R. (2001). *Knowledge in Action: Logical Foundations for Specifying and Implementing Dynamical Systems*. Cambridge: The MIT Press.

Steinsvold, C. (2008). A note on logics of ignorance and borders. *Notre Dame Journal of Formal Logic*, **49**(4), 385–392.

van der Hoek, W., & Lomuscio, A. (2003). Ignore at your peril - towards a logic for ignorance. In *Proceedings of 2nd AAMAS*. New York: ACM, pp. 1148–1149.

van der Hoek, W., & Lomuscio, A. (2004). A logic for ignorance. *Electronic Notes in Theoretical Computer Science*, **85**(2), 117–133.

van Ditmarsch, H. (2007). Comments to 'Logics of public communications'. *Synthese*, **158**(2), 181–187.

van Ditmarsch, H., Fan, J., van der Hoek, W., & Iliev, P. (2014). Some exponential lower bounds on formula-size in modal logic. *Advances in Modal Logic*, Vol. 10. pp. 139–157.

van Ditmarsch, H., van der Hoek, W., & Kooi, B. (2007). *Dynamic Epistemic Logic*, Vol. 337 of Synthese Library. Dordrecht: Springer.

Wang, Y., & Cao, Q. (2013). On axiomatizations of public announcement logic. *Synthese*, **190**(1S), 103–134.

Wang, Y., & Fan, J. (2013). Knowing that, knowing what, and public communication: Public announcement logic with Kv operators. In *Proceedings of 23rd IJCAI*, Palo Alto: AAAI Press, pp. 1147–1154.

Wang, Y., & Fan, J. (2014). Conditionally knowing what. *Advances in Modal Logic*, Vol. 10. pp. 569–587.

Wang, Y., Sietsma, F., & van Eijck, J. (2011). Logic of information flow on communication channels. In Omicini, A., Sardina, S., and Vasconcelos, W., editors. *Declarative Agent Languages and Technologies VIII*, Vol. 6619 of *Lecture Notes in Computer Science*. Berlin, Heidelberg: Springer, pp. 130–147.

Zolin, E. (1999). Completeness and definability in the logic of noncontingency. *Notre Dame Journal of Formal Logic*, **40**(4), 533–547.

Zolin, E. (2001). Infinitary expressibility of necessity in terms of contingency. In Striegnitz, K., editor. *Proceedings of the Sixth ESSLLI Student Session*. pp. 325–334.

Zuber, R. (1982). Semantics restrictions on certains complementizers. In *Proceedings of the 13th International Congress of Linguists*, Tokyo, pp. 434-436.

JIE FAN, YANJING WANG
DEPARTMENT OF PHILOSOPHY
PEKING UNIVERSITY
BEIJING 100871, CHINA
E-mail: {fanjie, y.wang}@pku.edu.cn

HANS VAN DITMARSCH
LORIA-CNRS/UNIVERSITY OF LORRAINE
54506 VANDUVRE-LÈS-NANCY, FRANCE
E-mail: hans.van-ditmarsch@loria.fr